Task Semantics for Tense and Modality

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Tense Operators

Strict Model: Building on Kripke's semantics, Prior introduces $\mathcal{P} = \langle T, <, |\cdot| \rangle$ where:

1. T is a nonempty set of *times* (also called *moments* and *world-states*).
2. *Earlier than* $<$ is a strict partial order (possibly total or left-linear).
3. $|p_i| \subseteq T$ interprets each sentence letter $p_i \in \mathbb{L}$.

Strict History: A strict history $h_i = \langle T_i, <_i \rangle$ is a maximal total suborder of $\langle T, < \rangle$.

4. Let $H_{\mathcal{P}}$ be the set of all histories in $\mathcal{P}$.
5. Let $H^x_{\mathcal{P}} = \{h_i \in H_{\mathcal{P}} : x \in T_i\}$ be histories through time x.

Ockhamist Semantics: Thomason's (1970) reconstruction where $h_i \in H_{\mathcal{P}}$ and $x \in h_i$:

6. $\mathcal{P}, h_i, x \vDash p_i$ *iff* $x \in |p_i|$ and $x \in h_i$.
7. $\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{P}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y <_i x$ in history h_i.
8. $\mathcal{P}, h_i, x \vDash \boxed{\scriptstyle\mathrm{F}}\varphi$ *iff* $\mathcal{P}, h_i, y \vDash \varphi$ for all $y >_i x$ in history h_i.
9. $\mathcal{P}, h_i, x \vDash \boxdot\varphi$ *iff* $\mathcal{P}, h_j, x \vDash \varphi$ for all $h_j \in H^x_{\mathcal{P}}$.

Structural Problem: Times cannot occur more than once or in different orders.

10. Giving up irreflexivity/transitivity undermines the reading of $<$.

Two-Dimensional Semantics

Two-Dimensional Model: $\mathcal{M}_2 = \langle W, T, \leqslant, |\cdot| \rangle$ where W is a nonempty set of *worlds*, T is a nonempty set of *times*, $\leqslant$ is a weak total order, and $|p_i| \subseteq W \times T$.

Kaplan's Semantics: For world $w \in W$ and time $x \in T$:

1. $\mathcal{M}_2, w, x \vDash p_i$ *iff* $\langle w, x \rangle \in |p_i|$.
2. $\mathcal{M}_2, w, x \nvDash \bot$.
3. $\mathcal{M}_2, w, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}_2, w, x \nvDash \varphi$ or $\mathcal{M}_2, w, x \vDash \psi$.
4. $\mathcal{M}_2, w, x \vDash \boxed{\scriptstyle\mathrm{P}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y < x$.
5. $\mathcal{M}_2, w, x \vDash \boxed{\scriptstyle\mathrm{F}}\varphi$ *iff* $\mathcal{M}_2, w, y \vDash \varphi$ for all $y > x$.
6. $\mathcal{M}_2, w, x \vDash \Box\varphi$ *iff* $\mathcal{M}_2, u, x \vDash \varphi$ for all $u \in W$.

Logical Consequence: $\Gamma \vDash \varphi$ *iff* for any two-dimensional model $\mathcal{M}_2$, world $w \in W$, and time $x \in T$, if $\mathcal{M}_2, w, x \vDash \gamma$ for all premises $\gamma \in \Gamma$, then $\mathcal{M}_2, w, x \vDash \varphi$.

Invalid: $\nvDash \Box\varphi \to \triangle\varphi$ and $\nvDash \triangledown\varphi \to \Diamond\varphi$ (where $\triangle\varphi \coloneqq \boxed{\scriptstyle\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle\mathrm{F}}\varphi$ and $\triangledown\varphi \coloneqq \neg\triangle\neg\varphi$).

Task Semantics

World-States: W is a nonempty set of world-states (instantaneous configurations).

Temporal Order: $\mathcal{D} = \langle D, +, \leqslant \rangle$ is a totally ordered abelian group of *durations*.

Task Relation: '$w \Rightarrow_x u$' reads '$w \in W$ can transition to $u \in W$ in duration $x \in D$'.

Frame: $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ where the following principles hold:

1. $w \Rightarrow_0 w$ for all $w \in W$.
2. $w \Rightarrow_{x+y} v$ whenever $w \Rightarrow_x u$ and $u \Rightarrow_y v$.

World History: Any function $\tau : X \to W$ where:

3. $X \subseteq D$ is convex (if $x, z \in X$ and $x \leqslant y \leqslant z$, then $y \in X$).
4. Task constraint: $\tau(x) \Rightarrow_{y-x} \tau(y)$ for all $x, y \in X$ with $x \leqslant y$.

Model: $\mathcal{M} = \langle W, \mathcal{D}, \Rightarrow, |\cdot| \rangle$ where $\mathcal{F} = \langle W, \mathcal{D}, \Rightarrow \rangle$ is a frame and $|p_i| \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$ of the language.

Semantic Clauses: Letting $H_{\mathcal{F}}$ be the set of world histories, $\tau \in H_{\mathcal{F}}$, and time $x \in D$:

5. $\mathcal{M}, \tau, x \vDash p_i$ *iff* $x \in \mathtt{dom}(\tau)$ and $\tau(x) \in |p_i|$.
6. $\mathcal{M}, \tau, x \nvDash \bot$.
7. $\mathcal{M}, \tau, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ and $\mathcal{M}, \tau, x \vDash \psi$.
8. $\mathcal{M}, \tau, x \vDash \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$.
9. $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y < x$.
10. $\mathcal{M}, \tau, x \vDash \boxed{\mathrm{F}}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for all $y > x$.

Valid: $\vDash \Box\varphi \to \triangle\varphi$ and $\vDash \triangledown\varphi \to \Diamond\varphi$.

Stability and Future Contingency

Stability: $\mathcal{M}, \tau, x \vDash \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x \vDash \varphi$ for all $\sigma \in H_{\mathcal{F}}$ where $\sigma(x) = \tau(x)$.

Will Always: $\boxed{!}\varphi := \boxdot\boxed{\mathrm{F}}\varphi$. *Could Always:* $\boxed{?}\varphi := \Diamond_{\cdot}\boxed{\mathrm{F}}\varphi$

Will Eventually: $\Diamond_{!}\varphi := \boxdot\Diamond_{\mathrm{F}}\varphi$ *Could Eventually:* $\Diamond_{?}\varphi := \Diamond_{\cdot}\Diamond_{\mathrm{F}}\varphi$

A Logic for Tense and Modality (TM)

TM is the smallest extension of propositional logic **PL** closed under:

MK If $\Gamma \vdash \varphi$, then $\Box\Gamma \vdash \Box\varphi$.

MB $\varphi \to \Box\Diamond\varphi$.

MT $\Box\varphi \to \varphi$.

M4 $\Box\varphi \to \Box\Box\varphi$.

TD If $\vdash \varphi$, then $\vdash \varphi_{\langle \mathrm{P}|\mathrm{F} \rangle}$.

MP $\varphi, \varphi \to \psi \vdash \psi$.

TF $\Box\varphi \to \boxed{\mathrm{F}}\Box\varphi$.

MF $\Box\varphi \to \Box\boxed{\mathrm{F}}\varphi$.

TK If $\Gamma \vdash \varphi$, then $\boxed{\mathrm{F}}\Gamma \vdash \boxed{\mathrm{F}}\varphi$.

TL $(\Diamond_{\mathrm{F}}\varphi \wedge \Diamond_{\mathrm{F}}\psi) \to \Diamond_{\mathrm{F}}(\varphi \wedge \psi) \vee \Diamond_{\mathrm{F}}(\Diamond_{\mathrm{F}}\varphi \wedge \psi) \vee \Diamond_{\mathrm{F}}(\varphi \wedge \Diamond_{\mathrm{F}}\psi)$.

T4 $\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi$.

TA $\varphi \to \boxed{\mathrm{F}}\Diamond_{\mathrm{P}}\varphi$.