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Logic Notes

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Propositional Logic: Syntax and Semantics

Canonical Name: A quoted symbol is the *canonical name* for the symbol quoted.

Language $\mathcal{L}$: The propositional language $\mathcal{L}$ includes: sentence letters 'P_1','P_2', ..., the sentential operators '$\vee$','$\neg$', and parentheses '(' and ')'.

Strings: The concatenation of a finite number of symbols in $\mathcal{L}$ is a *string* of $\mathcal{L}$.

Well Formed Sentences: Let 'A' and 'B' be *schematic variables* for strings of $\mathcal{L}$, and $\ulcorner\cdot\urcorner$ be a function from strings of $\mathcal{L}$ to the canonical names for those strings. We may then let $\mathcal{G}$ be the set of *well formed sentences* (wfs) of $\mathcal{L}$, which is defined recursively as follows:

- The sentence letters 'P_1','P_2',... are all wfs of $\mathcal{L}$.
- If A is a wfs of $\mathcal{L}$, then $\ulcorner\neg A\urcorner$ is a wfs of $\mathcal{L}$.
- If A and B are wfss of $\mathcal{L}$, then $\ulcorner(A \vee B)\urcorner$ is a wfs of $\mathcal{L}$.

Abbreviations: (i) $\ulcorner(A \wedge B)\urcorner$ abbreviates $\ulcorner\neg(\neg A \vee \neg B)\urcorner$;
(ii) $\ulcorner(A \to B)\urcorner$ abbreviates $\ulcorner(\neg A \vee B)\urcorner$;
(iii) $\ulcorner(A \leftrightarrow B)\urcorner$ abbreviates $\ulcorner[(A \to B) \wedge (B \to A)]\urcorner$.

Interpretation: Let $\mathcal{V}$ be an *interpretation* of $\mathcal{L}$ *iff* for every P_i of $\mathcal{L}$, either $\mathcal{V}(P_i) = T$ or $\mathcal{V}(P_i) = F$, but not both.

Semantics: Observe that $\mathcal{V}$ is only defined for the sentence letters $P_1, P_2, \ldots$ of $\mathcal{L}$ and not all other wfss. We may extend a given $\mathcal{V}$ to cover all wfss of $\mathcal{L}$ by recursively defining $\mathcal{V} \vDash A$, where '$\mathcal{V} \vDash A$' reads '$\mathcal{V}$ models A':

(P_i) $\mathcal{V} \vDash P_i$ *iff* $\mathcal{V}(P_i) = T$.
($\neg$) $\mathcal{V} \vDash \neg A$ *iff* it is not the case that $\mathcal{V} \vDash A$ (i.e., $\mathcal{V} \nvDash A$).[1]
($\vee$) $\mathcal{V} \vDash A \vee B$ *iff* $\mathcal{V} \vDash A$ or $\mathcal{V} \vDash B$.

Here, we use our grasp of the English terms 'it is not the case that' and 'or' to assign truth-values to all wfss of $\mathcal{L}$ on a given interpretation $\mathcal{V}$.

Logical Consequence: $\Gamma \vDash A$ *iff* for all interpretations $\mathcal{V}$, if $\mathcal{V} \vDash G$ for all $G \in \Gamma$, then $\mathcal{V} \vDash A$.

Logical Equivalence: $A \equiv B$ *iff* $A \vDash B$ and $B \vDash A$.

Logical Truth: A wfs A of $\mathcal{L}$ is *valid* (or a logical truth) iff $\varnothing \vDash A$ (written $\vDash A$).

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}^+$ include the symbols in $\mathcal{L}$ together with the sentential operators '$\wedge$', '$\to$', and '$\leftrightarrow$' which are to be read 'and', '(materially) implies that', and 'just in case', respectively.

(1) Provide a definition of the set $\mathcal{G}^+$ of wfss of $\mathcal{L}^+$.

(2) Provide a semantics for the wfss of $\mathcal{L}^+$.

(3) Prove $(A \wedge B)$, $(A \to B)$, and $(A \leftrightarrow B)$ are each logically equivalent to a wfs of $\mathcal{L}$.

(4) Provide two examples of logical truths including a sentential operator in $\mathcal{L}^+$.

[1] Strictly speaking, the schemata with schematic variables should be inclosed in corner quotes. It is common to also leave off corner quotes for sake of readability in defining the wfs of a language.

PROPOSITIONAL LOGIC: PROOF THEORY

Formal System: A *formal system* includes: (i) a language $\mathcal{L}$; (ii) a set $\mathcal{G}$ of the wfs of $\mathcal{L}$; (iii) a (possibly empty) set of axioms $\mathcal{A}$; and (iv) rules of inference $\mathcal{R}$ that permit the deduction of wfs from some other wfss.

Natural Deduction: A formal system is a *system of natural deduction* to the degree that its rules of inference closely resemble natural patterns of reasoning.

Rules of Inference: $\mathcal{R}^+$ includes the following rules, where the horizontal line indicates inference and brackets mark which assumptions are discharged.

Iteration:

| A
|---
| A (IT)

Assumption:

|---
| A (Asmp.)

Conjunction Introduction:

| A
| B
|---
| $A \wedge B$ ($\wedge$I)

Conjunction Elimination:

| $A \wedge B$
|---
| A ($\wedge$E)
| B ($\wedge$E)

Conditional Introduction:

| [A] (Asmp.)
| ⋮
| B
|---
| $A \rightarrow B$ ($\rightarrow$I)

Conditional Elimination:

| $A \rightarrow B$
| A
|---
| B ($\rightarrow$E)

Negation Introduction:

| [A] (Asmp.)
| B
| ⋮
| $\neg B$
|---
| $\neg A$ ($\neg$I)

Negation Elimination:

| $\neg\neg A$
|---
| A ($\neg$E)

Disjunction Introduction:

| A
|---
| $A \vee B$ ($\vee$I)
| $B \vee A$ ($\vee$I)

Disjunction Elimination:

| $A \vee B$
| [A] (Asmp.)
| ⋮
| C
| [B] (Asmp.)
| ⋮
| C
|---
| C ($\vee$E)

Problem Set: Translation and Deduction[2]

Translation: Resolve the following ambiguities (if any) by translating each into $\mathcal{L}^+$.

(1) Figaro exulted, and Basilio fretted, or the Court had a plan.

(2) Fred danced and sang or Ginger went home.

(3) If we are not in Paris then today is not Tuesday.

(4) The senator will not testify unless he is granted immunity.

(5) The senator will testify only if he is granted immunity.

(6) If Figaro does not expose the Count and force him to reform, then the Countess will discharge Susanna and resign to loneliness.

(7) The trade deficit will diminish and agriculture or industry will lead a recovery provided that both the dollar drops and neither Japan nor the EU raise their tariffs.

Arguments: Render the following arguments in the propositional language $\mathcal{L}^+$.

(1) Basilio fretted. Thus, if Figaro exulted, then Basilio fretted.

(2) Fred danced if Ginger went home. Fred didn't dance. And so Ginger didn't go home.

(3) If Figaro exulted, then the Court had a plan if Basilio fretted. Thus if Basilio fretted, then the Court had a plan if Figaro exulted.

(4) Fred danced or else Ginger sang and danced. It follows that either Fred danced or Ginger sang.

(5) If Lucy and Mary beat the record, then Paul will have to go. If Ian wins the race, then Paul can stay. Mary beat the record and Ian won the race. Therefore Lucy did not beat the record.

(6) If we are in Paris, then we are in Paris.

(7) It is not the case that we both are, and are not in Paris.

(8) Either Ginger or Fred danced. But Fred did not dance. Thus Ginger must have been the one who danced.

(9) Basilio fretted or Gigaro exulted. If Basilio fretted, the Court had a plan. But Gigaro did not exult, if David did not save the day. And so either the Court had a plan, or David saved the day.

(10) Kant is out for a walk just in case it is half noon. So either Kant is out for a walk and it is half noon, or Kant is not out for a walk and it is not half noon.

(11) It is not the case that Fred either sang or danced. It follows that Fred did not sing, nor did he dance.

(12) It is not the case that Fred sang and danced. It follows that Fred did not sing, or else did he did not dance.

(13) If we are in Paris, then we are in France. We are not in France. So we are not in Paris.

(14) If we are in Paris, then we are in France. If we are in France, we are in Europe. It follows that if we are in Paris, we are in Europe.

Deduction: Use the natural deduction rules $\mathcal{R}^+$ to prove that the conclusion of each of the regimented arguments above follows from its premises.

[2] I have adapted the following problems from Goldfarb (2003) and Laboreo (2005).

PROPOSITIONAL LOGIC: METALOGIC

Proof: Given a formal system $\mathcal{F} = \langle \mathcal{L}, \mathcal{G}, \mathcal{A}, \mathcal{R} \rangle$, a $\mathcal{F}$-*proof* of a conclusion A from a set of premises Γ is a finite sequence of $\mathcal{G}$-wfs of $\mathcal{L}$ such that every undischarged sentence in the proof is either: (i) a member of Γ; (ii) an axiom in $\mathcal{A}$; or else (iii) follows from the preceding sentences in the sequence by a rule of inference in $\mathcal{R}$.

Deduction: $\Gamma \vdash_{\mathcal{F}} A$ just in case there is a $\mathcal{F}$-proof of A from Γ.

Theorem: A *theorem* of a formal system $\mathcal{F}$ is a wfs of that system provable from no premises, i.e., any wfs A of $\mathcal{L}$ for which $\varnothing \vdash_{\mathcal{F}} A$ (written $\vdash_{\mathcal{F}} A$).

Logic: The *logic* of a formal system $\mathcal{F}$ is the set of all of its theorems.

Soundness: A formal system is *sound* just in case all of its theorems are valid.

Completeness: A formal system is *complete* just in case every valid wfs is a theorem.

Propositional Logic: The formal system of natural deduction $\mathcal{F}^+ = \langle \mathcal{L}^+, \mathcal{G}^+, \mathcal{A}^+, \mathcal{R}^+ \rangle$ given above is sound and complete, where $\mathcal{A}^+ = \varnothing$.

Equivalence: Two formal systems with the same logic may be said to be *equivalent*. There are many equivalent formal systems for propositional logic, i.e., systems with exactly the same set of theorems.

FIRST-ORDER LOGIC: SYNTAX

Language $\mathcal{L}_1$: The first-order language $\mathcal{L}_1$ includes: constants 'c_1', 'c_2', …, variables 'x_1', 'x_2', …, n-place predicates 'P_1^n', 'P_2^n', …, for each natural number $n \geqslant 0$, sentential operators '$\vee$','$\neg$', '$\exists x_i$', and parentheses '(' and ')'.

Terms: A symbol is a *term* just in case that symbol is a constant or variable.

Well Formed Formulas: Let 't_1',…, 't_n' be terms of $\mathcal{L}_1$, 'x' be a variable of $\mathcal{L}_1$, 'H^n' be an n-place predicate of $\mathcal{L}_1$, and 'A' and 'B' name arbitrary sentences of $\mathcal{L}_1$. We may then let $\mathcal{G}_1$ be the set of wff of $\mathcal{L}_1$, defined recursively as follows:

- The 0-place predicates 'P_1^0','P_2^0',… are all wff of $\mathcal{L}_1$.
- If H^n is an n-place predicate of $\mathcal{L}_1$, and $t_1, \ldots, t_n$ are terms of $\mathcal{L}_1$, then the atomic sentence $\ulcorner H^n(t_1, \ldots, t_n) \urcorner$ is a wff of $\mathcal{L}_1$.
- If A is a wff of $\mathcal{L}_1$, then $\ulcorner \neg A \urcorner$ is a wff of $\mathcal{L}_1$.
- If A and B are wffs of $\mathcal{L}_1$, then $\ulcorner (A \vee B) \urcorner$ is a wff of $\mathcal{L}_1$.
- If A is a wff of $\mathcal{L}_1$, then $\ulcorner \forall x A \urcorner$ is a wff of $\mathcal{L}_1$.

Abbreviations: (i) $\ulcorner (A \wedge B) \urcorner$ abbreviates $\ulcorner \neg(\neg A \vee \neg B) \urcorner$;
(ii) $\ulcorner (A \rightarrow B) \urcorner$ abbreviates $\ulcorner (\neg A \vee B) \urcorner$;
(iii) $\ulcorner (A \leftrightarrow B) \urcorner$ abbreviates $\ulcorner [(A \rightarrow B) \wedge (B \rightarrow A)] \urcorner$;
(iv) $\ulcorner \exists x A \urcorner$ abbreviates $\ulcorner \neg \forall x \neg A \urcorner$.

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}_1^+$ include the symbols in $\mathcal{L}_1$ together with the sentential operators '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', and '$\exists x_i$' which are to be read 'and', '(materially) implies that', 'just in case', and 'every x_i is such that', respectively. Provide a definition $\mathcal{G}_1^+$ of the wfss of $\mathcal{L}_1^+$.

First-Order Logic: Proof Theory

Free Variable: Every variable which occurs in an atomic sentence of $\mathcal{L}_1$ is *free*. If x is free in the wff A, then x is *bound* in the wff $\exists x A$. The wfss of $\mathcal{L}_1$ are those wff of $\mathcal{L}_1$ with no free variables.

Substitution: For any wfs A and terms t and k, let $\ulcorner A(t/k) \urcorner$ be the wfs which result from replacing every occurrence of k in the wfs A with t.

Available: A term t is *available* (written $t^\star$) for substitution in A iff t does not occur in A or in any premise or undischarged assumption used to prove A.

Rules of Inference: Let $\mathcal{R}_1^+$ extend $\mathcal{R}^+$ to also include the following rules of inference:

Universal Introduction:

$$\begin{array}{|l} A(t^\star/x) \\ \hline \forall x A(x) \end{array} \qquad (\forall I)$$

Universal Elimination:

$$\begin{array}{|l} \forall x A(x) \\ \hline A(t/x) \end{array} \qquad (\forall E)$$

Existential Introduction:

$$\begin{array}{|l} A(t/x) \\ \hline \exists x A(x) \end{array} \qquad (\exists I)$$

Existential Elimination:

$$\begin{array}{|l} \exists x A(x) \\ \hline A(t^\star/x) \end{array} \qquad (\exists E)$$

First-Order Logic: Semantics

Domain: Let the *domain* $\mathcal{D}$ be a set of objects.

Cartesian Domain: Let $\mathcal{D}^n$ be the set of all ordered tuples $\langle d_1, \ldots, d_n \rangle$ where each d_i is an object in the domain $\mathcal{D}$, i.e., $\mathcal{D}^n = \{\langle d_1, \ldots, d_n \rangle : d_i \in \mathcal{D} \text{ for } 1 \leqslant i \leqslant n\}$.

Interpretation: Let $\mathcal{V}_1$ be an *interpretation* of $\mathcal{L}_1$ over $\mathcal{D}$ just in case: (i) $\mathcal{V}_1(P_i^n) \subseteq \mathcal{D}^n$ for every $i \geqslant 1$ and $n \geqslant 0$; and (ii) $\mathcal{V}_1(c_i) \in \mathcal{D}$ for every $i \geqslant 1$.

Assignment: An *assignment* $\underline{a}$ is a function from the variables in $\mathcal{L}_1$ to the members of $\mathcal{D}$ such that $\underline{a}(x_i)$ is a member of the domain $\mathcal{D}$ for every $i \geqslant 1$.

Denotation: Let $I(t) = \begin{cases} \mathcal{V}_1(t) & \text{if } t = c_i \text{ for any } i \geqslant 1 \\ \underline{a}(t) & \text{if } t = x_i \text{ for any } i \geqslant 1 \end{cases}$

Variant: The function $\underline{a}[d/x]$ is an *x-variant* of the assignment $\underline{a}$ just in case $\underline{a}[d/x]$ differs from $\underline{a}$ at most by setting $\underline{a}[d/x](x) = d$.

Model: A *model* of $\mathcal{L}_1$ is any ordered pair $\mathcal{M} = \langle \mathcal{D}, \mathcal{V}_1 \rangle$, where $\mathcal{D}$ is a domain of individuals, and $\mathcal{V}_1$ an interpretation over $\mathcal{D}$.

Semantics: Given a model $\mathcal{M}$ of $\mathcal{L}_1$, and assignment $\underline{a}$, we may recursively define $\mathcal{M}, \underline{a} \vDash A$ for all wfss A of $\mathcal{L}_1$ as follows:

(P_i) $\mathcal{M}, \underline{a} \vDash P_i^n(t_1, \ldots, t_n)$ *iff* $\langle I(t_1), \ldots, I(t_n) \rangle \in \mathcal{V}_1(P_i^n)$.

($\exists$) $\mathcal{M}, \underline{a} \vDash \exists x_i A$ *iff* $\mathcal{M}, \underline{a}[d/x_i] \vDash A$, for some $d \in \mathcal{D}$.

($\neg$) $\mathcal{M}, \underline{a} \vDash \neg A$ *iff* $\mathcal{M}, \underline{a} \nvDash A$.

($\vee$) $\mathcal{M}, \underline{a} \vDash A \vee B$ *iff* $\mathcal{M}, \underline{a} \vDash A$ or $\mathcal{M}, \underline{a} \vDash B$.

It is important that in the case where $n = 0$, we adopt the convention that $\mathcal{V}_1(P_i^0) = \{\varnothing\}$ indicates truth, and $\mathcal{V}_1(P_i^0) = \varnothing$ indicates falsity.

First-Order Logic: Metalogic

Truth on a Model: $\mathcal{M} \vDash_1 A$ *iff* $\mathcal{M}, \underline{a} \vDash A$ for all variable assignments $\underline{a}$.

Logical Consequence: $\Gamma \vDash_1 A$ *iff* for all models $\mathcal{M}$, if $\mathcal{M} \vDash G$ for all $G \in \Gamma$, then $\mathcal{M} \vDash A$.

Logical Equivalence: $A \equiv_1 B$ *iff* $A \vDash_1 B$ and $B \vDash_1 A$.

Logical Truth: A wfs A of $\mathcal{L}_1$ is *valid* (or a logical truth) just in case $\vDash_1 A$.

First-Order Logic: The first-order formal system of natural deduction $\mathcal{F}_1^+ = \langle \mathcal{L}_1^+, \mathcal{G}_1^+, \mathcal{A}_1^+, \mathcal{R}_1^+ \rangle$ is sound and complete, where $\mathcal{A}_1^+ = \varnothing$.

Problem Set: First-Order Logic[3]

Semantics: Provide a semantics for the wfss of $\mathcal{L}_1^+$.

Translation: Translate the following arguments into $\mathcal{L}_1^+$.

(1) Everything that is beautiful is beautiful.

(2) Every philosopher is happy. So if everything is a philosopher, everything is happy.

(3) Everything is a philosopher and everything is happy. It follows that everything is a happy philosopher.

(4) Something is such that it is happy if Ella is a philosopher. So if Ella is a philosopher, then something is happy.

(5) There is a beautiful country. And so something is beautiful and something is a country.

(6) Nothing is ugly, and so everything is not ugly.

(7) Something is not right. It follows that not everything is right.

(8) Not everything is free. And so something is not free.

(9) Everything is not free. It follows that nothing is free.

(10) Every philosopher is wise, and everything wise is happy. Thus, every philosopher is happy.

(11) Every philosopher is happy. There is a wise philosopher. And something is wise and happy.

(12) Everything loves everything. Thus, everything loves itself.

(13) Something loves itself. And so something loves something.

(14) Nothing loves something which returns its loves.

Deduction: Use the natural deduction rules $\mathcal{R}_1^+$ to prove that the conclusion of each of the regimented arguments above follows from its premises.

Metalogic: Prove that every theorem of $\mathcal{F}^+$ is also a theorem of $\mathcal{F}_1^+$.

Bonus: Translate the following into $\mathcal{L}_1^+$:

(1) Everybody loves somebody.

(2) Everybody everybody loves loves somebody.

(3) Everybody everybody everybody loves loves loves somebody.

(4) You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

[3] I have adapted some of the following problems from Carr (2013). See also Halbach (2010).

PROPOSITIONAL MODAL LOGIC: MOTIVATION

Paradox: Substitution instances of the following schemata are theorems of $\mathcal{F}^+$:

(1) $A \to (B \to A)$. (2) $\neg A \to (A \to B)$.

But intuitively, a true proposition is not implied by any proposition whatsoever, nor does a false proposition imply any proposition.

Examples:
- If sugar is sweet, then if roses are red, sugar is sweet.
- If snow is not green, then if snow is green, roses are red.

Problem: The material conditional '$\to$' fails to adequately capture a strong enough sense of 'implies', sometimes represented in natural language by means of conditional constructions such as 'if..., then...'.

Desiderata: Lewis (1912) and Lewis and Langford (1932) developed modal logic in attempt to better capture the "usual sense" of 'implies'.

PROPOSITIONAL MODAL LOGIC: SYNTAX

Language $\mathcal{L}_\Box$: The propositional language $\mathcal{L}_\Box$ includes: sentence letters 'P_1','P_2', ..., the sentential operators '$\vee$','$\neg$', '$\Box$', and parentheses '(' and ')'.

Well Formed Sentences: Let 'A' and 'B' name arbitrary sentences of $\mathcal{L}$. We may then let $\mathcal{G}_\Box$ be the set of wfs of $\mathcal{L}_\Box$, defined recursively as follows:

- The sentence letters 'P_1','P_2',... are all wfs of $\mathcal{L}_\Box$.
- If A is a wfs of $\mathcal{L}_\Box$, then $\ulcorner \Box A \urcorner$ is a wfs of $\mathcal{L}_\Box$.
- If A is a wfs of $\mathcal{L}_\Box$, then $\ulcorner \neg A \urcorner$ is a wfs of $\mathcal{L}_\Box$.
- If A and B are wfss of $\mathcal{L}_\Box$, then $\ulcorner (A \vee B) \urcorner$ is a wfs of $\mathcal{L}_\Box$.

Abbreviations: (i) $\ulcorner (A \wedge B) \urcorner$ abbreviates $\ulcorner \neg(\neg A \vee \neg B) \urcorner$;
(ii) $\ulcorner (A \to B) \urcorner$ abbreviates $\ulcorner (\neg A \vee B) \urcorner$;
(iii) $\ulcorner (A \leftrightarrow B) \urcorner$ abbreviates $\ulcorner [(A \to B) \wedge (B \to A)] \urcorner$;
(iv) $\ulcorner \Diamond A \urcorner$ abbreviates $\ulcorner \neg \Box \neg A \urcorner$.

Strict Conditional: Lewis and Langford (1932) took the *strict conditional* '$\prec$' to better approximate the "usual sense" of 'implies', where '$A \prec B$' abbreviates '$\Box(A \to B)$'. It is typical to maintain the latter as standard notation.

Problem Set: Motivation and Translation

Motivation: Prove that the paradoxes of the material conditional (1) and (2) given above are theorems of $\mathcal{F}^+$.

Abbreviation: Let $\mathcal{L}_\Box^+$ include the symbols in $\mathcal{L}_\Box$ as well as '$\wedge$', '$\to$', '$\leftrightarrow$', and '$\Diamond$' which are read 'and', '(materially) implies that', 'just in case', and 'possibly', respectively. Provide a definition of $\mathcal{G}_\Box^+$ which includes all and only the wfss of $\mathcal{L}_\Box^+$ where $\mathcal{G}_\Box \subseteq \mathcal{G}_\Box^+$.

Translation: Translate the following into $\mathcal{L}_\Box^+$ as well as $\mathcal{L}_\Box$.

(1) It could rain or it could not rain.
(2) If it is necessary that it rains, then it is necessary that it could rain.
(3) It is necessary that it could either rain or not.

PROPOSITIONAL MODAL LOGIC: SEMANTICS

Frame: A *Kripke frame* $\mathcal{K}$ is an ordered pair $\langle W, R\rangle$, where W is a set of points called *possible worlds*, R is an *accessibility* relation between worlds.

Interpretation: $\mathcal{V}_{\Box}$ is an *interpretation* of $\mathcal{L}_{\Box}$ over W just in case for each $w \in W$ and $i \geqslant 1$, either $\mathcal{V}_{\Box}(P_i)(w) = 1$ or $\mathcal{V}_{\Box}(P_i)(w) = 0$, but not both.

Model: A *model* of $\mathcal{L}_{\Box}$ is any ordered triple $\mathcal{M}_{\Box} = \langle W, R, \mathcal{V}_{\Box}\rangle$ where $\langle W, R\rangle$ is a Kripke frame and $\mathcal{V}_{\Box}$ is an interpretation of $\mathcal{L}_{\Box}$.

Semantics: Given a model $\mathcal{M}_{\Box}$ of $\mathcal{L}_{\Box}$, and a world $w \in W$, we may recursively define $\mathcal{M}_{\Box}, w \vDash A$ for all wfss A of $\mathcal{L}_{\Box}$ as follows:

(P_i) $\mathcal{M}_{\Box}, w \vDash P_i$ *iff* $\mathcal{V}_{\Box}(P_i)(w) = 1$.

($\Box$) $\mathcal{M}_{\Box}, w \vDash \Box A$ *iff* $\mathcal{M}_{\Box}, w' \vDash A$ for every $w' \in W$ such that $R(w, w')$.

($\neg$) $\mathcal{M}_{\Box}, w \vDash \neg A$ *iff* $\mathcal{M}, \underline{a} \nvDash A$.

($\vee$) $\mathcal{M}_{\Box}, w \vDash A \vee B$ *iff* $\mathcal{M}, \underline{a} \vDash A$ or $\mathcal{M}, \underline{a} \vDash B$.

Proposition: The proposition $[\![A]\!]_{\mathcal{M}_{\Box}}$ that a wfs A of $\mathcal{L}_{\Box}$ expresses on a model $\mathcal{M}_{\Box}$ is the set of worlds $\{w \in W : \ \mathcal{M}_{\Box}, w \vDash A\}$ at which A is true. Every model $\mathcal{M}_{\Box}$ of $\mathcal{L}_{\Box}$ may then be though of as assigning each wfs of $\mathcal{L}_{\Box}$ to a proposition, conceived of as a subset of W.

PROPOSITIONAL MODAL LOGIC: AXIOMATIC SYSTEMS

Axioms: Consider the following axiom schemata and frame constraints:

(K)	$\Box(A \to B) \to (\Box A \to \Box B)$.	*None.*
(T)	$\Box A \to A$.	$R(w, w)$.
(B)	$A \to \Box\Diamond A$.	$R(w, w') \to R(w', w)$.
(4)	$\Box A \to \Box\Box A$.	$[R(w, w') \wedge R(w', w'')] \to R(w, w'')$.

Rules of Inference: Let $\mathcal{R}_{\Box}^{+}$ include the following rules of inference:

Necessitation:

$$\begin{array}{|l} A \\ \hline \Box A \end{array} \qquad \text{(N)}$$

Modus Ponens:

$$\begin{array}{|l} A \to B \\ A \\ \hline B \end{array} \qquad \text{(MP)}$$

Universal Substitution:

$$\begin{array}{|l} A \\ \hline A[B/C] \end{array} \qquad \text{(US)}$$

$A[B/C]$ is the result of replacing all occurrences of C in A with B, where B and C are wfs of $\mathcal{L}_{\Box}$.

Systems:

(K) The formal system $K = \langle \mathcal{L}_{\Box}^{+}, \mathcal{G}_{\Box}^{+}, \mathcal{A}_{K}^{+}, \mathcal{R}_{\Box}^{+}\rangle$, where $\mathcal{A}_{K}^{+}$ includes the theorems of $\mathcal{F}^{+}$ together with all instances of K.

(T) The formal system $T = \langle \mathcal{L}_{\Box}^{+}, \mathcal{G}_{\Box}^{+}, \mathcal{A}_{T}^{+}, \mathcal{R}_{\Box}^{+}\rangle$, where $\mathcal{A}_{T}^{+}$ includes the theorems of $\mathcal{F}^{+}$ together with all instances of K and T.

($S4$) The formal system $S4 = \langle \mathcal{L}_{\Box}^{+}, \mathcal{G}_{\Box}^{+}, \mathcal{A}_{4}^{+}, \mathcal{R}_{\Box}^{+}\rangle$, where $\mathcal{A}_{4}^{+}$ includes the theorems of $\mathcal{F}^{+}$ together with all instances of K, T, and 4.

($S5$) The formal system $S5 = \langle \mathcal{L}_{\Box}^{+}, \mathcal{G}_{\Box}^{+}, \mathcal{A}_{5}^{+}, \mathcal{R}_{\Box}^{+}\rangle$, where $\mathcal{A}_{5}^{+}$ includes the theorems of $\mathcal{F}^{+}$ together with all instances of K, T, B, and 4.

Problem Set: Axiomatic Proofs[4]

Credence: Evaluate the plausibility of each of the modal axioms when '$\Box$' and '$\Diamond$' are read as metaphysical necessity and possibility, respectively.

Translation: Translate the axioms belonging to $\mathcal{A}_5^+$ into natural language.

Proofs: Provide a proof of each of the following:

(1) $\vdash_K \Box(P \to Q) \to \Box(\neg Q \to \neg P)$.
(2) $\vdash_K (\Box P \wedge \Box Q) \to \Box(P \to Q)$.
(3) $\vdash_T \Box P \to \Diamond P$.
(4) $\vdash_T \neg\Box(P \wedge \neg P)$.
(5) $\vdash_{S4} \Box P \to \Box\Diamond\Box P$.
(6) $\vdash_{S4} \Diamond\Diamond\Diamond P \to \Diamond P$.
(7) $\vdash_{S5} \Diamond(P \wedge \Diamond Q) \leftrightarrow (\Diamond P \wedge \Diamond Q)$.

PROPOSITIONAL MODAL LOGIC: METALOGIC

Truth on a Model: $\mathcal{M} \vDash A$ *iff* $\mathcal{M}, w \vDash A$ for all $w \in W$.

Logical Consequence: $\Gamma \vDash_{\mathcal{C}} A$ *iff* for all $\mathcal{M} \in \mathcal{C}$, if $\mathcal{M} \vDash G$ for all $G \in \Gamma$, then $\mathcal{M} \vDash A$.

Logical Equivalence: $A \equiv_{\mathcal{C}} B$ *iff* $A \vDash_{\mathcal{C}} B$ and $B \vDash_{\mathcal{C}} A$.

Logical Truth: A wfs A of $\mathcal{L}_\Box$ is *valid* on a class of models $\mathcal{C}$ just in case $\vDash_{\mathcal{C}} A$.

Reflexive: A frame $\mathcal{K} = \langle W, R\rangle$ is *reflexive* just in case $R(w, w)$ for every $w \in W$. A model $\mathcal{M}_\Box = \langle W, R, \mathcal{V}_\Box\rangle$ is *reflexive* just in case $\langle W, R\rangle$ is a reflexive frame. Let $\mathcal{C}_r$ be the class of all reflexive models of $\mathcal{L}_\Box$.

Symmetric: A frame $\mathcal{K} = \langle W, R\rangle$ is *symmetric* just in case $R(w', w)$ whenever $R(w, w')$. A model $\mathcal{M}_\Box = \langle W, R, \mathcal{V}_\Box\rangle$ is *symmetric* just in case $\langle W, R\rangle$ is symmetric. Let $\mathcal{C}_s$ be the class of all symmetric models of $\mathcal{L}_\Box$.

Transitive: A frame $\mathcal{K} = \langle W, R\rangle$ is *transitive* just in case $R(w, w'')$ whenever $R(w, w')$ and $R(w', w'')$. A model $\mathcal{M}_\Box = \langle W, R, \mathcal{V}_\Box\rangle$ is *transitive* just in case $\langle W, R\rangle$ is transitive. Let $\mathcal{C}_t$ be the class of transitive models of $\mathcal{L}_\Box$.

Modal Logics:

(K) The modal system K is sound and complete over the class of all models $\mathcal{C}_K$, i.e., $\vdash_K A$ if and only if $\vDash_{\mathcal{C}_K} A$.

(T) The modal system T is sound and complete over the class of all reflexive models $\mathcal{C}_T = \mathcal{C}_r$, i.e., $\vdash_T A$ if and only if $\vDash_{\mathcal{C}_T} A$.

($S4$) The system $S4$ is sound and complete over the reflexive and transitive models $\mathcal{C}_{S4} = \mathcal{C}_r \bigcap \mathcal{C}_t$, i.e., $\vdash_{S4} A$ if and only if $\vDash_{\mathcal{C}_{S4}} A$.[5]

($S5$) The modal system $S5$ is sound and complete over the class of all reflexive, symmetric, and transitive models $\mathcal{C}_{S5} = \mathcal{C}_r \bigcap \mathcal{C}_s \bigcap \mathcal{C}_t$, i.e., $\vdash_{S5} A$ if and only if $\vDash_{\mathcal{C}_{S5}} A$.[6]

Counter Model: A *counter model* for a wfs A of $\mathcal{L}_\Box$ is a model of $\mathcal{L}_\Box$ in which A is false.

Invalidity: To demonstrate that a wfs A of $\mathcal{L}_\Box$ is *invalid* on the class of models $\mathcal{C}$ (i.e., $\nvDash_{\mathcal{C}} A$), it is sufficient to specify a single counter model to A in $\mathcal{C}$.

[4] I have adapted some of the following exercises from Studd (2016) and Sider (2010).

[5] The intersection $X \bigcap Y$ is the set of elements in both X and Y, i.e., $X \bigcap Y = \{z : z \in X \text{ and } z \in Y\}$.

[6] See Hughes and Cresswell (1996) for proofs of soundness and completeness for $K, T, S4$, and $S5$.

Problem Set: Further Exercises

Semantic Proofs: Give semantic arguments to demonstrate each of the following:

(1) $\vDash_{\mathcal{C}_T} \Box A \to A$.
(2) $\vDash_{\mathcal{C}_T} \Box A \to \Box A$.
(3) $\vDash_{\mathcal{C}_{S4}} \Box\Box A \to \Box A$.
(4) $\vDash_{\mathcal{C}_{S4}} \Box A \to \Box\Box A$.
(5) $\vDash_{\mathcal{C}_{S5}} A \to \Box\Box A$.
(6) $\vDash_{\mathcal{C}_{S5}} \Box A \to \Box\Box A$.

Equivalences: Provide semantic proofs of the following equivalences:

(7) $\neg\Box A \equiv_{\mathcal{C}_K} \Box\neg A$.
(8) $\neg\Box A \equiv_{\mathcal{C}_K} \Box\neg A$.
(9) $\neg\Box\neg \equiv_{\mathcal{C}_K} \Box A$.
(10) $\neg\Box\neg \equiv_{\mathcal{C}_K} \Box A$.

Counter Models: Provide counter models to demonstrate the following:

(11) $\nvDash_{\mathcal{C}_K} \Box A \to A$.
(12) $\nvDash_{\mathcal{C}_K} \Box A \to \Box A$.
(13) $\nvDash_{\mathcal{C}_{S4}} A \to \Box\Box A$.
(14) $\nvDash_{\mathcal{C}_{S4}} \Box A \to \Box\Box A$.
(15) $\nvDash_{\mathcal{C}_T} \Box A \to \Box\Box A$.
(16) $\nvDash_{\mathcal{C}_T} \Box A \to \Box\Box A$.

Propositions: Draw on the semantic definitions above to establish the following:

(17) $\mathcal{M}_\Box, w \vDash A$ *iff* $w \in [\![A]\!]_{\mathcal{M}_\Box}$.
(18) $\mathcal{M}_\Box, w \vDash A \to B$ *iff* $w \in [\![A]\!]^c_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.[7]
(19) $\mathcal{M}_\Box, w \vDash \Box(A \to B)$ *iff* $[\![A]\!]_{\mathcal{M}_\Box} \subseteq [\![B]\!]_{\mathcal{M}_\Box}$.
(20) $\mathcal{M}_\Box, w \vDash \Box(A \leftrightarrow B)$ *iff* $[\![A]\!]_{\mathcal{M}_\Box} = [\![B]\!]_{\mathcal{M}_\Box}$.
(21) $[\![\neg A]\!]_{\mathcal{M}_\Box} = [\![A]\!]^c_{\mathcal{M}_\Box}$.[8]
(22) $[\![A \wedge B]\!]_{\mathcal{M}_\Box} = [\![A]\!]_{\mathcal{M}_\Box} \bigcap [\![B]\!]_{\mathcal{M}_\Box}$.
(23) $[\![A \vee B]\!]_{\mathcal{M}_\Box} = [\![A]\!]_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.
(24) $[\![A \to B]\!]_{\mathcal{M}_\Box} = [\![A]\!]^c_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.

Paradoxes: Prove the following analogues of the paradoxes given above:

(25) $\Box A \to \Box(B \to A)$.
(26) $\neg\Box A \to \Box(A \to B)$.

Irrelevance: Prove the following for an arbitrary A, B and $\mathcal{M}_\Box$ of $\mathcal{L}_\Box$:

(27) If $\Box A$, then $[\![B]\!]_{\mathcal{M}_\Box} = [\![B \wedge A]\!]_{\mathcal{M}_\Box}$.
(28) If $\neg\Box A$, then $[\![B]\!]_{\mathcal{M}_\Box} = [\![B \vee A]\!]_{\mathcal{M}_\Box}$.

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[7] The union $X \bigcup Y$ is the set of elements in both X and Y, i.e., $X \bigcup Y = \{z : z \in X \text{ or } z \in Y\}$.
[8] The complement X^c is the set of elements in W that are not in X, i.e., $X^c = \{z \in W : z \notin X\}$.