A Complete Logic of Ground

Benjamin Brast-McKie

January 4, 2022

Abstract

Drawing on Kit Fine's (2012c, 2017b,c) state semantics, this chapter establishes soundness and completeness for *The Specific Logic of Unilateral Ground* (UGS). I will begin by restricting consideration in §2 to a proof system UGSN in which a specificity operator has been included and negation does not occur within the scope of a grounding operator, providing a semantics for UGSN in §3 where propositions are closed under finite fusion. After proving soundness and completeness for UGSN in §4 and §5, will extend these results in §6 to a class of models where propositions are closed under infinite fusion. In §7, I will extend UGSN to UGS which permits negation to occur within the scope of the unilateral grounding operator, showing that UGS is sound and complete over the class of bilateral infinite fusion models. By contrast with the Boolean lattices of extensional and intensional logics, I will show in §8 that the space of hyperintensional propositions under study form a bounded bilattice which is neither distributive nor interlaced. I will conclude in §9 by defining bilateral essence and ground in terms of unilateral ground, deriving a range of theorems and admissible rules by which to reason with bilateral essence and ground.

1 Introduction

In a number of publications, Kit Fine (2012c, 2016, 2017a,b,c) has developed a hyperintensional theory of propositions in which propositions are *exactly verified* and *exactly falsified* by *states*.[1] Whereas worlds are understood to be maximal on account of determining the truth-value of every proposition whatsoever, states are required to be wholly relevant to the propositions that they exactly verify or falsify, and so are able to draw distinctions which worlds cannot. In addition to abandoning the maximality required of worlds, states need not be possible. Although a primitive distinction between possible and impossible states plays a critical role in developing a state semantics for modal logic, no such distinction will be needed for the present pursuit. Accordingly, both the semantic machinery used to study validity in the logic of ground, as

[1] Fine has developed applications of the state semantic framework to a wide variety of topics, reaching well beyond the semantics of ground. See Fine (2012a, 2014, 2013, 2018a,b, 2020).

well as the intended model used to guide our interpretation of the semantics may be said to be free of any dependence on primitive modality.

Given the requirement that states be wholly relevant to the propositions which they exactly verify or falsify, the exact verifiers for A and $\neg A$ will not typically partition the set of states. Rather, many states may exactly verify neither A nor $\neg A$, where similarly, many states may exactly falsify neither A nor $\neg A$. Moreover, the exact verifiers for A will in no way determine the exact verifiers for $\neg A$, and so without recourse to exact falsification in addition to exact verification, there is no systematic way to relate the propositions expressed by a sentence and its negations. Accordingly, I will follow Fine in taking the exact verifiers for $\neg A$ to be the exact falsifiers for A, and the exact falsifiers for $\neg A$ to be the exact verifiers for A. In this way the *bilateral proposition* consisting of both exact verifiers and falsifiers assigned to a sentence will determine the bilateral proposition assigned to its negation.[2]

Just as the set of exact verifiers (falsifiers) for $\neg A$ is not the complement within the set of states of the exact falsifiers (verifiers) A, it also follows from the requirement that states be wholly relevant to the propositions which they exactly verify or falsify that conjunction cannot be interpreted by set intersection as in extensional and intensional semantics. For instance, the exact verifiers for A may be disjoint from the exact verifiers for B, and yet $A \wedge B$ may have exact verifiers. Given a notion of state fusion, I will follow Fine in taking $A \wedge B$ to be exactly verified by any fusion of an exact verifier for A and an exact verifier for B. Similarly, given an exact falsifier for A, and an exact falsifier for B, their fusion will be an exact falsifier for $A \vee B$. Additionally, I will assume what Fine calls an *inclusive semantics* for disjunction and conjunction where any exact verifier for A, B, or $A \wedge B$ is an exact verifier for $A \vee B$, and any exact falsifier for A, B, or $A \vee B$ is an exact falsifier for $A \wedge B$. As brought out below, the sets of exact verifiers and falsifiers for propositions will be closed under fusion. Building on Fine's work, it remains to develop a logic for the most natural entailment relations holding between propositions in this hyperintensional setting.

We may begin by observing that in any theory of propositions where propositional identity satisfies the Boolean identities, disjunctive part may be shown to be the converse of conjunctive part, where entailment may be defined in terms of either. More specifically, letting $A \leqslant B \coloneqq A \vee B \equiv B$ and $A \sqsubseteq B \coloneqq A \wedge B \equiv B$ where '$\equiv$' is taken to express propositional identity in a given theory, we may show that if the theory of propositions in question is Boolean, then $(A \leqslant B) \leftrightarrow (B \sqsubseteq A)$.[3] For instance, assuming an extensional theory of propositions, '$\leqslant$' expresses the material conditional and '$\sqsubseteq$' expresses its converse. Similarly, in an intensional theory of propositions, '$\leqslant$' expresses the strict conditional whereas '$\sqsubseteq$' expresses its converse. On both extensional and intensional theories of propositions, there is just one entailment relation

[2] See Fine (2017d,b) for further discussion.

[3] *Proof:* If $A \leqslant B$, it follows that $A \vee B \equiv B$, and so by substitution $A \wedge (A \vee B) \equiv A \wedge B$. By absorption, $A \equiv A \wedge B$, and so $B \sqsubseteq A$. The reverse derivation is similar. □

since nothing new is added by also including converse relations.

Whereas in extensional and intensional logics, conjunctive and disjunctive part are converse relations, these definitions correspond to distinct entailment relations in the present setting. As brought out in §8, the theory of bilateral propositions elaborated below does not satisfy the Boolean absorption laws, where as a result $(A \leqslant B) \leftrightarrow (B \sqsubseteq A)$ does not hold in general. Accordingly, the state semantic framework admits of two natural entailment relations $\sqsubseteq$ and $\leqslant$ which I will call by the names *essence* and *ground*, respectively. Informally, I will take '$\sqsubseteq$' and '$\leqslant$' to regiment 'necessary for' and 'sufficient for' respectively, where essence and ground may be shown to track relevance in addition to modal profile. Additionally, I will take both essence and ground to be "worldly" insofar as they are to relate propositions understood as *things being certain ways* rather than representations of things being some way or other. Moreover, I will assume essence and ground are reflexive and non-factive so that every proposition grounds itself independent of whether that proposition obtains or not. Both the irreflexive and factive analogues may then be defined in terms of the reflexive non-factive notions of essence and ground.[4]

Although essence and ground are not converses, essence and ground are nevertheless interdefinable in a language with a negation operator which satisfies the involution law $\neg\neg A \equiv A$. As shown in §8, both $(A \leqslant B) \leftrightarrow (\neg A \sqsubseteq \neg B)$ and $(A \sqsubseteq B) \leftrightarrow (\neg A \leqslant \neg B)$ are valid, where the space of bilateral propositions may be shown to form a bilattice ordered by essence and ground. Given that essence and ground are interdefinable, I will take $\ulcorner A \sqsubseteq B \urcorner$ to abbreviate $\ulcorner \neg A \leqslant \neg B \urcorner$ purely as a matter of convention, defining propositional identity in terms of essence and ground rather than *vice versa* in §9.[5] In order to articulate such definitions, negation must be permitted to occur within the scope of a grounding operator. However, we may observe that sentences of this kind are excluded from Fine's (2012c) *Pure Logic of Ground* (PLG) which aims to study the atomic sentences which can be articulated by a range of primitive grounding operators in the absence of any other operators.[6]

Instead of admitting distinct primitives for each notion of ground that Fine includes in PLG while excluding all other operators from the logic, I will begin with a single primitive grounding operator '$\trianglelefteq$' which I will refer to as *unilateral ground*. In contrast to $\leqslant$ which imposes constraints on both the exact verifiers and falsifiers for the propositions involved, $A \trianglelefteq B$ only requires that every exact verifier for A is also an exact verifier for B. We may observe that the semantics for unilateral ground is similar to the semantics for the material and strict conditionals, only that bilateral propositions have been substituted for extensional and intensional propositions. Nevertheless, we may define $\leqslant$ and $\sqsubseteq$ in terms of $\trianglelefteq$ given a language which includes the extensional operators:

Unilateral Equivalence: Let $\ulcorner A \approx B \urcorner$ abbreviate $\ulcorner (A \leqslant B) \wedge (B \leqslant A) \urcorner$.

[4] See Chapter 2 as well as Fine (2001, 2012b, 2015) for related philosophical discussion.

[5] See §5 in Chapter 2 and §1 in Chapter 4 for related discussion of these definitions.

[6] See Chapter 1 for further discussion.

Unilateral Essence: Let $\ulcorner A \trianglerighteq B \urcorner$ abbreviate $\ulcorner A \wedge B \approx B \urcorner$.

Ground: Let $\ulcorner A \leqslant B \urcorner$ abbreviate $\ulcorner (A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B) \urcorner$.

Essence: Let $\ulcorner A \sqsubseteq B \urcorner$ abbreviate $\ulcorner (A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B) \urcorner$.

Restricting attention to a language which includes '$\trianglelefteq$' rather than '$\leqslant$' among its primitive operators has the effect of greatly simplifying the present attempt to provide a complete logic for ground.[7] In order to define ground in terms of unilateral ground, the operator for negation must be permitted to occur within the scope of a grounding operator, where conjunction is permitted to occur between atomic grounding sentences. Accordingly, I will lift the restrictions that Fine imposes in developing PLG, permitting any extensional combination of atomic unilateral grounding sentences of the form $\ulcorner A \trianglelefteq B \urcorner$ where at most the extensional operators may occur within $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$.

Despite admitting of a much wider range of well-formed sentences than is included in PLG, I will nevertheless exclude consideration of sentences in which grounding operators occur within the scope of a grounding operator, as well as purely extensional sentences and extensional combinations on purely extensional and non-extensional sentences such as $\ulcorner A \wedge (A \leqslant B) \urcorner$. Although one might hope to provide a logic which admits of a wider range of sentences, such ambitions reach beyond the scope of the present pursuit.[8]

By contrast with the attempt to establish completeness for a logic with a primitive grounding operator, Fine and Jago (2019) provide a complete logic for their *exact entailment* relation $\Vdash$ which holds between a set of sentences in a purely extensional language and a further sentence of that language. Whereas grounding operates on propositions, exact entailment is a logical consequence relation which quantifies over both models and states.[9] Fine and Jago also require the exact verifiers and falsifiers for a proposition to be *convex* insofar as any state between two exact verifiers (falsifiers) in mereological order for a proposition must also be an exact verifier (falsifier) for that proposition. However, imposing convexity requires making some alteration to the inclusive semantics for conjunction and disjunction.[10] Nevertheless, were Fine and Jago to abandon their convexity constraint, we could show that $A \Vdash B$ just in case $A \trianglelefteq B$ is valid on the definition of validity that I will go on to provide. Thus I will not further consider exact entailment in what follows.

In addition to taking '$\trianglelefteq$' to be primitive and '$\leqslant$' to be defined, I will also include a *specificity operator* '\$' in the language presented below. Let a proposition be called *specific* just in case exactly one state verifies that proposition.[11] The specificity operator may be compared to the *singularity*

[7] See Chapter 1 and Fine (2012b) for discussion of a range of further grounding operators which may be defined in terms of ground in a language which includes extensional operators.

[8] See §7 of Chapter 4 for discussion of the semantics limitations the present framework faces in attempting to interpret sentences with nested grounding operators.

[9] I have adapted their notation to avoid confusion with what is to come below.

[10] See §6 in Chapter 3 and §3 in Chapter 4 for discussion.

[11] Fine (2017c, p. 695) employs the label 'determinate' rather than 'specific'.

predicate '$\texttt{S}$' included in plural logics where ⌜$\texttt{S}(aa)$⌝ reads ⌜There is exactly one of the aas⌝, as well as to the *numerical possibility operator* '$\Box_1$' where ⌜$\Box_1 A$⌝ reads ⌜There is exactly one possibility in which A⌝. As brought out in §5, the specificity operator will play a crucial role in the completeness proof, where states in the Henkin model are ground-theoretic equivalence classes generated from specific sentences, where attempts to establish completeness without a specificity operator faced numerous difficulties. Although the motivation for including the specificity operator in the logic takes a purely technical form, specificity operators are nevertheless of interest in their own right.

The completeness proof will be structured as follows. I will begin in §2 by introducing *The Specific Logic of Unilateral Ground without Negation* (UGSN) in which negation does not occur within the scope of a unilateral grounding operator, providing a state semantics for UGSN in which the exact verifiers for a proposition are closed under finite fusion in §3. I will then prove that UGSN is both sound and complete over the class $\mathcal{C}$ of finite fusion models in §4 and §5, respectively. These results will then be extended to a class of models $\mathcal{C}^\infty$ in which exact verifiers are also closed under infinite fusion in §6, proving in §7 that *The Specific Logic of Unilateral Ground* (UGS) in which negation is permitted to occur within the scope of a unilateral grounding operator is sound and complete over the class $\mathcal{C}^\pm$ of bilateral infinite fusion models. Instead of forming a Boolean lattice, I will I show in §8 that the space of propositions forms a bounded bilattice which is neither distributive nor interlaced. I will conclude in §9 by deriving a subsystem of UGS which I will call *The Logic of Essence and Ground* (EG) which includes a range of theorems and admissible rules for reasoning with essence and ground.

2 Proof Theory

We may begin by restricting consideration to a simplified propositional language $\mathcal{L}^-$ where the well formed sentences are built up in two separate stages. Given a set of sentence letters $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$ together with a set of *extremal constants* $\mathbb{E} = \{\mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\perp\}$, we may define the *pre-formed sentences* (pfs) of $\mathcal{L}^-$ as follows, where $p \in \mathbb{L}$ and $e \in \mathbb{E}$ are both arbitrary:

$$A ::= p \mid e \mid A \wedge A \mid A \vee A.$$

I will refer to '$\mathcal{T}$' as the *top* and '$\bot$' as the *bottom*, reading '$\mathcal{V}$' as the *verum* and '$\perp\!\!\!\perp$' as the *falsum*. As brought out in §3, $\mathcal{T}$ is the proposition which is exactly verified by any state whatsoever, whereas $\mathcal{V}$ is only exactly verified by the fusion of all states. Additionally, $\bot$ is exactly verified by no states, whereas $\perp\!\!\!\perp$ is only exactly verified by the state which trivially obtains.

Let $\texttt{pfs}(\mathcal{L}^-)$ be the set of pfs of $\mathcal{L}^-$, where $A, B, \ldots \in \texttt{pfs}(\mathcal{L}^-)$, and $\texttt{comp}(A)$ is the number of occurrences of '$\vee$' and '$\wedge$' in A. If $A, B \in \texttt{pfs}(\mathcal{L}^-)$, then both ⌜$\$A$⌝ and ⌜$A \trianglelefteq B$⌝ are *well formed atomic sentence* (wfas) of $\mathcal{L}^-$, where $\texttt{atoms}(\mathcal{L}^-)$ is the set of all wfas of $\mathcal{L}^-$. We may then define the *well* *formed sentences* (wfs) of $\mathcal{L}^-$, letting $\alpha \in \texttt{atoms}(\mathcal{L}^-)$ be arbitrary:

$$\varphi ::= \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Let $\texttt{wfs}(\mathcal{L}^-)$ be the set of wfs of $\mathcal{L}^-$, where $\varphi, \psi, \ldots \in \texttt{wfs}(\mathcal{L}^-)$, and let $\texttt{comp}^+(\varphi)$ be the number of occurrences of '$\neg$', '$\vee$', and '$\wedge$' in φ which do not occur in any preformed subsentences (sub-pfs) of φ. For any sentences $A, B \in \texttt{pfs}(\mathcal{L}^-)$, $\ulcorner A \trianglelefteq B \urcorner$ reads ⌜It being the case that A grounds it being the case that B⌝, or more simply, ⌜A grounds B⌝, indicating that A obtaining is sufficient for B to obtain. We may also read ⌜$\$A$⌝ as ⌜There is exactly one way for it to be the case that A⌝, or more simply, ⌜A is specific⌝, indicating that there is only one state which exactly verifies A. As we will see, the theorems of the logic belong to $\texttt{wfs}(\mathcal{L}^-)$ and not to $\texttt{pfs}(\mathcal{L}^-)$.

As usual, '$\rightarrow$' and '$\leftrightarrow$' may be introduced as metalinguistic abbreviations, where I will also take $\ulcorner A \ntrianglelefteq B \urcorner$ to abbreviate $\ulcorner \neg(A \trianglelefteq B) \urcorner$, letting $\ulcorner A \approx B \urcorner$ abbreviate $\ulcorner (A \trianglelefteq B) \wedge (B \trianglelefteq A) \urcorner$ which expresses ground-theoretic equivalence. Given the formation rules for $\mathcal{L}^-$, we may define syntactic consequence $\vdash_{\text{UGSN}}$ for *The Specific Logic of Unilateral Ground without Negation* (UGSN) to be the smallest relation closed under truth-functional consequence which satisfies the following, where $A, B, C, D \in \texttt{pfs}(\mathbb{L})$ and $\Gamma \cup \{\varphi\} \subseteq \texttt{wfs}(\mathbb{L})$:

Grounding Axioms and Rules

GA1 $A \trianglelefteq A \vee B$.

GA2 $B \trianglelefteq A \vee B$.

GA3 $A \trianglelefteq A \wedge A$.

GA4 $A \wedge A \trianglelefteq A$.

GA5 $A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$.

GA6 $(A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$.

GA7 $A \wedge B \trianglelefteq B \wedge A$.

GA8 $A \trianglelefteq B, C \trianglelefteq D \vdash A \wedge C \trianglelefteq B \wedge D$.

GA9 $A \trianglelefteq B, B \trianglelefteq C \vdash A \trianglelefteq C$.

Extremal Axioms

VF1 $\$\perp$.

VF2 $\$\mathcal{V}$.

VF3 $\perp \wedge A \trianglelefteq A$.

VF4 $A \trianglelefteq \perp \wedge A$.

VF5 $\mathcal{V} \wedge A \trianglelefteq \mathcal{V}$.

VF6 $\mathcal{V} \trianglelefteq \mathcal{V} \wedge A$.

VF7 $A \trianglelefteq \mathcal{T}$.

Specificity Rules

SP1 $\$A \vdash A \ntrianglelefteq \perp$.

SP2 $A \approx B \vdash \$A \leftrightarrow \$B$.

SP3 $\$A, \$B \vdash \$(A \wedge B)$.

SP4 $\$A, B \trianglelefteq A \vdash (A \trianglelefteq B) \vee (B \trianglelefteq \perp)$.

SP5 $\$A, A \trianglelefteq C \vee D \vdash (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$.

SP6 If $\Gamma \vdash \$p \rightarrow [(p \trianglelefteq A) \rightarrow (p \trianglelefteq B)]$ where $p \in \mathbb{L}$ does not occur in A, B, or in any $\gamma \in \Gamma$, then $\Gamma \vdash A \trianglelefteq B$.

SP7 If $\Gamma, [(A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)] \vdash \varphi$ where distinct $p, q \in \mathbb{L}$ do not occur in φ or in any $\gamma \in \Gamma$, then $\Gamma \vdash \varphi$.

A wfs φ of $\mathcal{L}^-$ is a *theorem* of UGSN just in case $\vdash_{\text{UGSN}} \varphi$. Letting $\ulcorner \varphi_{[A/B]} \urcorner$ be the result of uniformly replacing all occurrences of $\ulcorner B \urcorner$ in $\ulcorner \varphi \urcorner$ with $\ulcorner A \urcorner$, and taking $\ulcorner \varphi_{(A/B)} \urcorner$ to be the result of replacing zero or more occurrences of $\ulcorner B \urcorner$ in $\ulcorner \varphi \urcorner$ with $\ulcorner A \urcorner$, we may derive the following admissible rules and theorems:

Admissible Rules

AR1 $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$.

AR2 If $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$.

AR3 $A \approx B \vdash_{\text{UGSN}} C \approx C_{(A/B)}$.

AR4 $A \approx B, \varphi \vdash_{\text{UGSN}} \varphi_{(A/B)}$.

Grounding Theorems

T1 $A \wedge \bot \trianglelefteq B$.

T2 $A \trianglelefteq B \vee \mathcal{T}$.

T3 $A \trianglelefteq A \wedge (A \vee B)$.

T4 $A \trianglelefteq A \vee (A \wedge B)$.

T5 $A \wedge B \trianglelefteq A \vee B$.

T6 $A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C)$.

Equivalence Theorems

E1 $A \approx A$.

E2 $A \wedge \bot \approx \bot$.

E3 $A \vee \mathcal{T} \approx \mathcal{T}$.

E4 $A \wedge \mathcal{V} \approx \mathcal{V}$

E5 $A \vee \bot \approx A$.

E6 $A \wedge \top \approx A$.

E7 $A \vee A \approx A$.

E8 $A \wedge A \approx A$.

E9 $A \vee B \approx B \vee A$.

E10 $A \wedge B \approx B \wedge A$.

E11 $A \vee (B \vee C) \approx (A \vee B) \vee C$.

E12 $A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$.

E13 $A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$.

E14 $A \wedge (B \wedge C) \approx (A \wedge B) \wedge (A \wedge C)$.

It is worth observing that of the equivalences for bounded distributive lattices, the following four principles are conspicuously absent:

#Abs1 $A \wedge (A \vee B) \approx A$.

#Dist $A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$.

#Abs2 $A \vee (A \wedge B) \approx A$.

The following section will present countermodels to each of the above, and so the corresponding space of propositions does not form a distributive lattice. We may also note that although **E13** is a theorems of UGSN, its dual **#Dist** is not. As I will bring out in §7, these apparent asymmetries may be shown to be an artefact of our present concern with unilateral propositions. Once negation is permitted to occur within the scope of the grounding operator, we may derive a range of theorems in §8 which maintains duality.

I will now provide proofs for selected theorems and admissible rules which are novel to UGSN, assuming standard results from classical logic.

AR1 $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$.

Proof. Let $A, B, C \in \texttt{pfs}(\mathcal{L}^-)$ and choose some $p \in \mathbb{L}$ which does not occur in A, B, or C. Thus $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \wedge B \trianglelefteq C$ follows from **GA8** and **GA4** by **GA9**. Additionally, we know by **SP5** that $\$p, p \trianglelefteq A \vee B \vdash_{\text{UGSN}} (p \trianglelefteq A) \vee (p \trianglelefteq B) \vee (p \trianglelefteq A \wedge B)$. Consider:

$$p \trianglelefteq A, A \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{1}$$

$$p \trianglelefteq B, B \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{2}$$

$$p \trianglelefteq A \wedge B, A \wedge B \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{3}$$

It follows that $A \trianglelefteq C, B \trianglelefteq C, \$p, p \trianglelefteq A \vee B \vdash_{\text{UGSN}} p \trianglelefteq C$. Thus we know that $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \vee B) \to (p \trianglelefteq C)]$, and so by **SP6** $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$ since p does not occur in A, B, or C. □

AR2 If $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$. (*Uniform Substitution*)

Proof. The proof goes by induction on the number of applications of the metarules in UGSN, where the only novel cases are given below:

Case SP6: Assume $\Gamma \vdash^n_{\text{UGSN}} \varphi$ follows by **SP6**. Thus $\varphi = A \trianglelefteq B$ where $\Gamma \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A) \to (q \trianglelefteq B)]$ for some $q \in \mathbb{L}$ which does not occur in A, B, or any $\gamma \in \Gamma$. Thus we know by hypothesis that $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q_{[A/p]} \to [(q_{[A/p]} \trianglelefteq A_{[A/p]}) \to (q_{[A/p]} \trianglelefteq B_{[A/p]})]$. If $p = q$, then it follows trivially that $\Gamma_{[A/p]} \vdash^n_{\text{UGSN}} \varphi_{[A/p]}$ since q does not occur in A, B, or any $\gamma \in \Gamma$. If instead $p \neq q$, then it follows from the above that $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A_{[A/p]}) \to (q \trianglelefteq B_{[A/p]})]$, for some $q \in \mathbb{L}$ which does not occur in A, B, or in any $\gamma \in \Gamma$. Again by hypothesis, we know that $\Gamma_{[A/p],[q^\star/q]} \vdash^{n-1}_{\text{UGSN}} \$q_{[q^\star/q]} \to [(q_{[q^\star/q]} \trianglelefteq A_{[A/p]}) \to (q_{[q^\star/q]} \trianglelefteq B_{[A/p]})]$, where we may choose $q^\star \in \mathbb{L}$ to be the sentence letter with the lowest index which does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Equivalently, $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q^\star \to [(q^\star \trianglelefteq A_{[A/p]}) \to (q^\star \trianglelefteq B_{[A/p]})]$ where $q^\star$ does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Thus it follows by **SP6** that $\Gamma_{[A/p]} \vdash^n_{\text{UGSN}} A_{[A/p]} \trianglelefteq B_{[A/p]}$, and so $\Gamma_{[A/p]} \vdash^n_{\text{UGSN}} \varphi_{[A/p]}$.

Case SP7: Assume $\Gamma \vdash^n_{\text{UGSN}} \varphi$ follows by **SP7**. Thus it follows that $\Gamma, [\$A \wedge (A \trianglelefteq B \wedge C)] \to [\$r \wedge \$q \wedge (r \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (r \wedge q \trianglelefteq A)] \vdash_{\text{UGSN}} \varphi$ where distinct $r, q \in \mathbb{L}$ do not occur in φ. By hypothesis we know that:

$$\Gamma_{[A/p]}, [\$A_{[A/p]} \wedge (A_{[A/p]} \trianglelefteq B_{[A/p]} \wedge C_{[A/p]})] \to$$
$$[\$r_{[A/p]} \wedge \$q_{[A/p]} \wedge (r_{[A/p]} \trianglelefteq B_{[A/p]}) \wedge (q_{[A/p]} \trianglelefteq C_{[A/p]}) \wedge (r_{[A/p]} \wedge q_{[A/p]} \trianglelefteq A_{[A/p]})]$$
$$\vdash^{n-1}_{\text{UGSN}} \varphi_{[A/p]}$$

$\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q_{[A/p]} \to [(q_{[A/p]} \trianglelefteq A_{[A/p]}) \to (q_{[A/p]} \trianglelefteq B_{[A/p]})]$. If $p = q$, then it follows trivially that $\Gamma_{[A/p]} \vdash^n_{\text{UGSN}} \varphi_{[A/p]}$ since q does not occur in A, B, or any $\gamma \in \Gamma$. If instead $p \neq q$, then it follows from the above that $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A_{[A/p]}) \to (q \trianglelefteq B_{[A/p]})]$, for some $q \in \mathbb{L}$ which does not occur in A, B, or in any $\gamma \in \Gamma$. Again by hypothesis, we know that $\Gamma_{[A/p],[q^\star/q]} \vdash^{n-1}_{\text{UGSN}} \$q_{[q^\star/q]} \to [(q_{[q^\star/q]} \trianglelefteq A_{[A/p]}) \to (q_{[q^\star/q]} \trianglelefteq B_{[A/p]})]$, where we may choose $q^\star \in \mathbb{L}$ to be the sentence letter with the lowest index which does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Equivalently, $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q^\star \to [(q^\star \trianglelefteq A_{[A/p]}) \to (q^\star \trianglelefteq B_{[A/p]})]$ where $q^\star$ does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Thus it follows by **SP7** that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} A_{[A/p]} \trianglelefteq B_{[A/p]}$, and so $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$.

Given that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$ holds in each of the cases above, we may conclude by discharge that if $\Gamma \vdash^{n}_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$, and so by induction that if $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$. □

T1 $\vdash_{\text{UGSN}} A \wedge \bot \trianglelefteq B$.

Proof. Let $A, B \in \texttt{pfs}(\mathcal{L}^-)$ and choose some $p \in \mathbb{L}$ where p does not occur in either A or B. Observe that $\$p \vdash_{\text{UGSN}} p \ntrianglelefteq \bot$ by **SP1**, and so by propositional logic we know that $\$p, p \trianglelefteq A \wedge \bot \vdash_{\text{UGSN}} (p \ntrianglelefteq \bot) \wedge (p \trianglelefteq A \wedge \bot)$. Choose distinct $r, q \in \mathbb{L}$ which do not occur in A, B, or p. Consider:

$$[(p \ntrianglelefteq \bot) \wedge (p \trianglelefteq A \wedge \bot)] \to [\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]. \quad (*)$$

Thus $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} \$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)$, follows by propositional logic. However, we know that $\vdash_{\text{UGSN}} \$r \to (r \ntrianglelefteq \bot)$ by **SP1**, and so $\vdash_{\text{UGSN}} \neg\$r \vee (r \ntrianglelefteq \bot)$ by abbreviation. It follows that $\vdash_{\text{UGSN}} \neg\$r \vee \neg\$q \vee (r \ntrianglelefteq \bot) \vee (q \ntrianglelefteq A) \vee (r \wedge q \ntrianglelefteq p)$ by propositional logic, and so $\vdash_{\text{UGSN}} \neg[\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]$. It follows that $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} \neg[\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]$. Thus $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} p \trianglelefteq B$ by *ex falso quodlibet*, and so we may conclude that $(*) \vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \wedge \bot) \to (p \trianglelefteq B)]$.

Given the above, $\vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \wedge \bot) \to (p \trianglelefteq B)]$ follows by **SP7**. Since $p \in \mathbb{L}$ where p does not occur in either A or B, we may conclude that $\vdash_{\text{UGSN}} A \wedge \bot \trianglelefteq B$ by **SP6** as desired. □

T6 $\vdash_{\text{UGSN}} A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C)$.

Proof. Let $A, B, C \in \texttt{pfs}(\mathcal{L}^-)$, and choose some $p \in \mathbb{L}$ where p does not occur in A, B, or C. Observe that $\$p \vdash_{\text{UGSN}} p \ntrianglelefteq \bot$ by **SP1**, and so by propositional logic $\$p, p \trianglelefteq A \wedge (B \vee C) \vdash_{\text{UGSN}} (p \ntrianglelefteq \bot) \wedge (p \trianglelefteq A \wedge (B \vee C))$. Choose distinct $r, q \in \mathbb{L}$ which do not occur in A, B, C, or p. Consider:

$$[(p \ntrianglelefteq \bot) \wedge (p \trianglelefteq A \wedge (B \vee C))] \to [\$r \wedge \$q \wedge (r \trianglelefteq A) \wedge (q \trianglelefteq B \vee C) \wedge (r \wedge q \trianglelefteq p)]. \quad (*)$$

So $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} \$r \wedge \$q \wedge (r \trianglelefteq A) \wedge (q \trianglelefteq B \vee C) \wedge (r \wedge q \trianglelefteq p)$, and so by **SP4** that $\$p, r \wedge q \trianglelefteq p \vdash_{\text{UGSN}} (p \trianglelefteq r \wedge q) \wedge (r \wedge q \trianglelefteq \bot)$, where it follows from **SP3** that $\$r, \$q \vdash_{\text{UGSN}} \$(r \wedge q)$, and from **SP1** that $\$(r \wedge q) \vdash_{\text{UGSN}} r \wedge q \ntrianglelefteq \bot$. Thus $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} p \trianglelefteq r \wedge q$ by propositional logic. We may now observe that it follows by **SP5** that $\$q, q \trianglelefteq B \vee C \vdash_{\text{UGSN}} (q \trianglelefteq B) \vee (q \trianglelefteq C) \vee (q \trianglelefteq B \wedge C)$. Consider:

$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq B \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C) \quad (1)$$
$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C) \quad (2)$$
$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq B \wedge C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C). \quad (3)$$

Here, (1) follows by **GA8**, **GA1**, and **GA9**, where (2) is similar but draws on **GA2** in place of **GA1**. In order to justify (3), it is enough to observe that $\vdash_{\text{UGSN}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge (A \wedge C)$ by **E14**, and so together with **T5** we may conclude that $\vdash_{\text{UGSN}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \vee (A \wedge C)$. Together with **GA8** and **GA9**, (3) follows. Thus we may conclude that $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} p \trianglelefteq (A \wedge B) \vee (A \wedge C)$. Since r and q are distinct and do not occur in A, B, C, or p, we know by **SP7** that $\$p, p \trianglelefteq A \wedge (B \vee C) \vdash_{\text{UGSN}} p \trianglelefteq (A \wedge B) \vee (A \wedge C)$, and so it follows that $\vdash_{\text{UGSN}} \$p \rightarrow [p \trianglelefteq A \wedge (B \vee C) \rightarrow p \trianglelefteq (A \wedge B) \vee (A \wedge C)]$. Thus $\vdash_{\text{UGSN}} A \wedge (B \vee C) \trianglelefteq (A \wedge B) \vee (A \wedge C)$ follows by **SP6**. □

E13 $\vdash_{\text{UGSN}} A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$.

Proof. Given that $\vdash_{\text{UGSN}} A \trianglelefteq A$ by **E1**, and $\vdash_{\text{UGSN}} B \trianglelefteq B \vee C$ by **GA1**, we know that $\vdash_{\text{UGSN}} A \wedge B \trianglelefteq A \wedge (B \vee C)$ follows by **GA8**. Similarly, $\vdash_{\text{UGSN}} A \wedge C \trianglelefteq A \wedge (B \vee C)$ follows by **E1**, **GA2**, and **GA8**. Thus $\vdash_{\text{UGSN}} (A \wedge B) \vee (A \wedge C) \trianglelefteq A \wedge (B \vee C)$ follows by **AR1**, and so together with **T6** we may conclude that $\vdash_{\text{UGSN}} A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$. □

3 Semantics

This section will draw on Kit Fine's (2017b,c) recent work in order to provide a semantics for $\mathcal{L}^-$. Rather than working over a space of possible worlds W with an accessibility relation R, I will follow Fine in taking S to be a set states of affairs, or *states* for short.[12] Let a *state space* $\mathcal{S}$ be any ordered pair $\langle S, \star \rangle$ where S is any set closed under the binary operation $\star$, mapping any two states to their *fusion*. A state space $\langle S, \star \rangle$ is *mereological* just in case it satisfies:

Null State: There is a *null state* $\square \in S$ such that $\square \star s = s$ for all $s \in S$.

Full State: There is a *full state* $\blacksquare \in S$ such that $\blacksquare \star s = \blacksquare$ for all $s \in S$.

Idempotency: $s \star s = s$ for all $s \in S$.

Commutativity: $s \star t = t \star s$ for all $s, t \in S$.

Associativity: $(s \star t) \star r = s \star (t \star r)$ for all $s, t, r \in S$.

Assuming a "no class"-theory of classes, we may let $\mathbb{M}$ be the class of all mereological state spaces, employing set notation where convenient. In order to define the relevant classes of models for $\mathcal{L}^-$, consider the following definitions:

Fusion: $\overline{X} = \{x \star y : x, y \in X\}$.

$\mathcal{S}$-Propositions: $\mathbb{P}_{\mathcal{S}} = \{X \subseteq S : X = \overline{X}\}$.

Given any $\mathcal{S} \in \mathbb{M}$ where $\mathcal{S} = \langle S, \star \rangle$, a *unilateral $\mathcal{S}$-model of $\mathcal{L}^-$* is an ordered triple $\mathcal{M}_v = \langle S, \star, |\cdot|_v \rangle$ where $|p|_v \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}$. For ease of exposition, it will often be convenient to drop the subscript which names the model. We

[12] See Fine (2017a, Draft) for discussion.

may then let $\mathcal{C}_{\mathcal{S}}$ be the set of all $\mathcal{S}$-models, and $\mathcal{C} = \bigcup\{\mathcal{C}_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$ be the proper class of all models of $\mathcal{L}^-$ whatsoever.

We may now provide a Finean state semantics for exact verification $\Vdash$ for all pfs $\mathcal{L}^-$ by means of the following recursive clauses:

Unilateral Pre-Semantics:

(p_i) $\mathcal{M}, s \Vdash p_i$ *iff* $s \in |p_i|$.

($\mathcal{T}$) $\mathcal{M}, s \Vdash \mathcal{T}$ *iff* $s = s$. ($\mathcal{V}$) $\mathcal{M}, s \Vdash \mathcal{V}$ *iff* $s = \blacksquare$.

($\bot$) $\mathcal{M}, s \Vdash \bot$ *iff* $s \neq s$. ($\perp\!\!\!\!\!\bot$) $\mathcal{M}, s \Vdash \perp\!\!\!\!\!\bot$ *iff* $s = \square$.

($\wedge$) $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = d \star t$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

($\vee$) $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

Since there is no threat of ambiguity in what follows, I will often drop 'exact' from 'exact verification'. The semantics may be called *unilateral* on account of only including clauses for verification as opposed to both verification and falsification as will be given below.[13] Whereas only the null state $\square$ verifies the $\perp\!\!\!\!\!\bot$, no state verifies the $\bot$. By contrast, every state verifies the $\mathcal{T}$, whereas only the full state $\blacksquare$ verifies $\mathcal{V}$. The conjunction clause formalises the idea that only a fusion of verifiers for each of the conjuncts will verify the conjunction as a whole. In the case of disjunction, a verifier for either disjunct will verify the disjunction, as will a fusion of verifiers for each of the disjuncts.

Given that each $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ assigns every $p \in \mathcal{L}^-$ to a proposition $|p| \in \mathbb{P}_{\mathcal{S}}$ consisting of the verifiers for p in $\mathcal{M}$, we may extend $|\cdot|$ to all $A \in \texttt{pfs}(\mathcal{L}^-)$:

Unilateral Valuation: $s \in |A|$ *iff* $\mathcal{M}, s \Vdash A$.

Letting $|A|$ be the proposition that $A \in \texttt{pfs}(\mathcal{L}^-)$ expresses in $\mathcal{M}$, we may define the *extremal propositions* in $\mathbb{P}_{\mathcal{S}}$ as follows:

Unilateral Extremal Propositions:

Top: $\mathcal{T}_{\mathcal{S}} = S$. *Verum:* $\mathcal{V}_{\mathcal{S}} = \{\blacksquare\}$.

Bottom: $\bot_{\mathcal{S}} = \varnothing$. *Falsum:* $\perp\!\!\!\!\!\bot_{\mathcal{S}} = \{\square\}$.

When ambiguity does not threaten, I will drop the subscript, letting context determine the corresponding state space.

Having extended the valuation function to all $A \in \texttt{pfs}(\mathcal{L}^-)$ in **Unilateral Valuation**, we may now state the semantics for the wfs of $\mathcal{L}^-$ as follows:

Semantics:

($\trianglelefteq$) $\mathcal{M} \vDash A \trianglelefteq B$ *iff* $|A| \subseteq |B_1|$.

[13] See Fine (2016, 2017a,b,c) for this usage. Fine (2017b) provides a semantics for sentences which includes negation, crediting Van Fraassen (1969) for providing a related construction.

($\$$) $\mathcal{M} \vDash \$A$ *iff* there is exactly one $s \in |A|$.

($\neg$) $\mathcal{M} \vDash \neg\varphi$ *iff* $\mathcal{M} \nvDash \varphi$.

($\wedge$) $\mathcal{M} \vDash \varphi \wedge \psi$ *iff* $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$.

($\vee$) $\mathcal{M} \vDash \varphi \vee \psi$ *iff* $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \psi$.

The semantic clause for grounding holds that $A \trianglelefteq B$ is true in $\mathcal{M}$ just in case every exact verifier for A is an exact verifier for B, where $\$A$ is true in $\mathcal{M}$ just in case A is verified by only one state.[14] Let φ be a *logical consequence* of Γ over $\mathcal{C}$— symbolised by '$\Gamma \vDash_{\mathcal{C}} \varphi$'— just in case for all $\mathcal{M} \in \mathcal{C}$, if $\mathcal{M} \vDash B$ for all $B \in \Gamma$, then $\mathcal{M} \vDash \varphi$. A wfs φ is $\mathcal{C}$-*valid* just in case $\vDash_{\mathcal{C}} \varphi$.

We may now define algebraic analogues of conjunction and disjunction which are defined over the space of propositions $\mathbb{P}_{\mathcal{S}}$:

Product: $X \wedge Y = \{x \star y : x \in X,\ y \in Y\}$.

Sum: $X \vee Y = X \cup Y \cup (X \wedge Y)$.

Given any $\mathcal{S} \in \mathbb{M}$, we may let $\mathcal{A}_{\mathcal{S}} = \langle \mathbb{P}_{\mathcal{S}}, \wedge, \vee, \bot, \perp\!\!\!\!\!-, \mathcal{V}, \mathcal{T} \rangle$ where not only are $\mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\!\!- \in \mathbb{P}_{\mathcal{S}}$, but we may show that $\mathbb{P}_{\mathcal{S}}$ is closed under $\wedge$ and $\vee$ as follows:

L3.2 $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

L3.3 $X \vee Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

By the formation rules, $\mathcal{A}_{\mathcal{L}^-} = \langle \texttt{pfs}(\mathcal{L}^-), \wedge, \vee, \mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\!\!- \rangle$ is an algebra with the same signature $\sigma_{\mathcal{L}^-} = \langle \{\wedge, \vee\}, \mathbb{E} \rangle$ as $\mathcal{A}_{\mathcal{S}}$, where the valuation function induced by any $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ is an $\mathcal{L}^-$-homomorphism $|\cdot| : \mathcal{A}_{\mathcal{L}^-} \to \mathcal{A}_{\mathcal{S}}$ as below:

$\mathcal{L}^-$-Homomorphism: For any $\mathcal{S} \in \mathbb{M}$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, the function $|\cdot| : \mathcal{L}^- \to \mathcal{A}_{\mathcal{S}}$ is an *$\mathcal{L}^-$-homomorphism iff* for every $A, B \in \texttt{pfs}(\mathbb{L})$ both: (1) $|A \wedge B| = |A| \wedge |B|$; and (2) $|A \vee B| = |A| \vee |B|$.

L3.4 $|A \wedge B| = |A| \wedge |B|$. **P3.1** $|A| \in \mathbb{P}_{\mathcal{S}}$.

L3.5 $|A \vee B| = |A| \vee |B|$.

The results above will be of use throughout what follows, where in §7 similar results may be shown to hold once negation is included in the language.

We may then observe that $\langle \mathbb{P}_{\mathcal{S}}, \wedge \rangle$ and $\langle \mathbb{P}_{\mathcal{S}}, \vee \rangle$ are semilattices on account of satisfying idempotency, commutativity, and associativity. In particular, the following identities hold for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$:

[14] Fine (2012b,c) gives a similar semantics for $\trianglelefteq$ in a language without $\$, \neg, \wedge$, and $\vee$.

L3.6 $X \wedge X = X$. **L3.9** $X \vee X = X$.

L3.7 $X \wedge Y = Y \wedge X$. **L3.10** $X \vee Y = Y \vee X$.

L3.8 $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$. **L3.11** $(X \vee Y) \vee Z = X \vee (Y \vee Z)$.

Despite these results, $\mathcal{A}_{\mathcal{S}}$ need not form a lattice on account of failing to satisfy the absorption laws, since there is some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$ such that:

L3.18 $X \wedge (X \vee Y) \neq X$. **L3.19** $X \vee (X \wedge Y) \neq X$.

In addition to failing to form a lattice, we may show that $\mathcal{A}_{\mathcal{S}}$ is non-distributive since there is some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$ such that:

L3.21 $X \vee (Y \wedge Z) \neq (X \vee Y) \wedge (X \vee Z)$.

These results correspond to the absence of **#Abs1**, **#Abs2**, and **#Dist** from UGSN, where the following section shows that **#Abs1**, **#Abs2**, and **#Dist** are not theorems of UGSN. Nevertheless, for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$:

L3.22 $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$.

Although $\vee$ does not distributes over $\wedge$, we may distribute $\wedge$ over $\vee$. As we will see in §7, the apparent asymmetry above will disappear once negation has been included in the language. The remainder of the present section will be devoted to establishing a selection of the claims enumerated above, drawing on these results to prove that UGSN is sound over $\mathcal{C}$ in the following section.

L3.1 $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$, letting $s \in X \wedge (Y \cup Z)$. It follows that $s = x \star u$ for some $x \in X$ and $u \in Y \cup Z$. If $u \in Y$, then $s \in X \wedge Y$, and so $s \in (X \wedge Y) \cup (X \wedge Z)$. Similarly, if $u \in Z$, then $s \in X \wedge Z$, and so $s \in (X \wedge Y) \cup (X \wedge Z)$. Thus $X \wedge (Y \cup Z) \subseteq (X \wedge Y) \cup (X \wedge Z)$.

Assume instead that $s \in (X \wedge Y) \cup (X \wedge Z)$. If $s \in X \wedge Y$, then $s = x \star y$ for some $x \in X$ and $y \in Y$, and so $y \in Y \cup Z$. Thus $s \in X \wedge (Y \cup Z)$. Similarly, if $s \in X \wedge Z$, then $s = x \star z$ for some $x \in X$ and $z \in Z$, and so $z \in Y \cup Z$. Thus $s \in X \wedge (Y \cup Z)$, and so $(X \wedge Y) \cup (X \wedge Z) \subseteq X \wedge (Y \cup Z)$. Together with the above, $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z)$ as needed. □

L3.2 $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ where $X, Y \in \mathbb{P}_{\mathcal{S}}$, and choose some $s \in X \wedge Y$. By *Idempotency*, $s \in \overline{X \wedge Y}$, and so $X \wedge Y \subseteq \overline{X \wedge Y}$. To establish the converse inclusion, assume that $s \in \overline{X \wedge Y}$. It follows that $s = x \star y$ for some $x, y \in X \wedge Y$, and so $x = u \star v$ and $y = w \star z$ for some $u, w \in X$ and $v, z \in Y$. It follows that $u \star w \in \overline{X}$ and $v \star z \in \overline{Y}$. Given that $X, Y \in \mathbb{P}_{\mathcal{S}}$, we know that $\overline{X} = X$ and $\overline{Y} = Y$, and so $u \star w \in X$ and $v \star z \in Y$. Thus $(u \star w) \star (v \star z) \in X \wedge Y$, where $(u \star w) \star (v \star z) = s$ by *Associativity* and *Commutativity*. It follows that $s \in X \wedge Y$, and so $\overline{X \wedge Y} \subseteq X \wedge Y$. Thus we may conclude that $X \wedge Y = \overline{X \wedge Y}$, and so $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$. □

L3.3 $X \vee Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Follows from the definition of $\mathbb{P}_{\mathcal{S}}$ together with the *Idempotency*, *Commutativity*, and *Associativity* of all $\mathcal{S} \in \mathbb{M}$ as in **L3.2**. □

L3.4 $|A \wedge B| = |A| \wedge |B|$ for all $A, B \in \texttt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$.

Proof. Let $A, B \in \texttt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$. By **Unilateral Valuation**, we know that $|A|, |B| \subseteq S$. We may then consider the following:

$$\begin{aligned} s \in |A \wedge B| \ \ &\textit{iff} \ \ \mathcal{M}, s \Vdash A \wedge B \\ &\textit{iff} \ \ s = a \star b \text{ where } \mathcal{M}, a \Vdash A \text{ and } \mathcal{M}, b \Vdash B \\ &\textit{iff} \ \ s = a \star b \text{ where } a \in |A| \text{ and } b \in |B| \\ &\textit{iff} \ \ s \in |A| \wedge |B|. \end{aligned}$$

All of the biconditionals above hold by definition. Thus we may conclude that $|A \wedge B| = |A| \wedge |B|$ as desired. □

L3.5 $|A \vee B| = |A| \vee |B|$ for all $A, B \in \texttt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$.

Proof. Let $A, B \in \texttt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$. By **Unilateral Valuation**, we know that $|A|, |B| \subseteq S$. We may then consider the following:

$$\begin{aligned} s \in |A \vee B| \ \ &\textit{iff} \ \ \mathcal{M}, s \Vdash A \vee B \\ &\textit{iff} \ \ \mathcal{M}, s \Vdash A, \text{ or } \mathcal{M}, s \Vdash B, \text{ or } \mathcal{M}, s \Vdash A \wedge B \\ &\textit{iff} \ \ s \in |A|, \text{ or } s \in |B|, \text{ or } s \in |A \wedge B| \\ (\dagger) \ \ &\textit{iff} \ \ s \in |A|, \text{ or } s \in |B|, \text{ or } s \in |A| \wedge |B| \\ &\textit{iff} \ \ s \in |A| \cup |B| \cup (|A| \wedge |B|) \\ (\ddagger) \ \ &\textit{iff} \ \ s \in |A| \vee |B|. \end{aligned}$$

The biconditionals above all hold by definition with the exception of (†) which follows by **L3.4**, and (‡) which follows by *Sum*. Thus we may conclude that $|A \vee B| = |A| \vee |B|$. □

L3.6 $X \wedge X = X$ for all $\mathcal{S} \in \mathbb{M}$ and $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X \in \mathbb{P}_{\mathcal{S}}$, and let $s \in X \wedge X$. It follows that $s = x \star x$ for some $x \in X$, and so $s \in \overline{X}$. Since $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$, we know that $\overline{X} = X$, and so $s \in X$. Thus $X \wedge X \subseteq X$.

Assume instead that $s \in X$. By definition, $s \star s \in X \wedge X$, and so $s \in X \wedge X$ by *Idempotency*. Thus $X \subseteq X \wedge X$, and so $X \wedge X = X$. □

L3.7 $X \wedge Y = Y \wedge X$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$, letting $s \in X \wedge Y$. It follows that $s = x \star y$ for some $x \in X$ and $y \in Y$. Thus $y \star x \in Y \wedge X$, and so $s \in Y \wedge X$ by *Commutativity*. We may then conclude that $X \wedge Y \subseteq Y \wedge X$, and so $X \wedge Y = Y \wedge X$ by symmetry of reasoning. □

L3.8 $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$, we may then reason as follows:

$$\begin{aligned} s \in (X \wedge Y) \wedge Z \quad & \textit{iff} \quad s = u \star z \text{ for some } u \in X \wedge Y \text{ and } z \in Z \\ & \textit{iff} \quad s = (x \star y) \star z \text{ for some } x \in X, y \in Y, \text{ and } z \in Z \\ (*) \quad & \textit{iff} \quad s = x \star (y \star z) \text{ for some } x \in X, y \in Y, \text{ and } z \in Z \\ & \textit{iff} \quad s = x \star v \text{ for some } x \in X, \text{ and } v \in Y \wedge Z \\ & \textit{iff} \quad s \in X \wedge (Y \wedge Z). \end{aligned}$$

The biconditionals above all hold by definition with the exception of $(*)$ which is given by *Associativity*. Thus $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$. ☐

L3.9 $X \vee X = X$ for all $\mathcal{S} \in \mathbb{M}$ where $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$.

Proof. By definition, $X \vee X = \overline{X \cup X} = \overline{X}$. Given that $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$, we know that $\overline{X} = X$, and so $X \vee X = X$ as desired. ☐

L3.10 $X \vee Y = Y \vee X$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. By definition, $X \vee Y = \overline{X \cup Y} = Y \vee X$ since $X \cup Y = Y \cup X$. ☐

L3.11 $(X \vee Y) \vee Z = X \vee (Y \vee Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$. We may then reason as follows:

$$\begin{aligned} (X \vee Y) \vee Z \quad & = \quad (X \vee Y) \cup Z \cup [(X \vee Y) \wedge Z] \\ & = \quad X \cup Y \cup Z \cup (X \wedge Y) \cup [(X \cup Y \cup [X \wedge Y]) \wedge Z] \\ (1) \quad & = \quad X \cup Y \cup Z \cup (X \wedge Y) \cup (X \wedge Z) \cup (Y \wedge Z) \cup [(X \wedge Y) \wedge Z] \\ (2) \quad & = \quad X \cup Y \cup Z \cup (Y \wedge Z) \cup (X \wedge Y) \cup (X \wedge Z) \cup [X \wedge (Y \wedge Z)] \\ (3) \quad & = \quad X \cup Y \cup Z \cup (Y \wedge Z) \cup [X \wedge (Y \cup Z \cup [Y \wedge Z])] \\ & = \quad X \cup (Y \vee Z) \cup [X \wedge (Y \vee Z)] \\ & = \quad X \vee (Y \vee Z). \end{aligned}$$

Whereas both (1) and (3) follow by **L3.1**, (2) is given by *Sum*. Thus we may conclude that $(X \vee Y) \vee Z = X \vee (Y \vee Z)$, as needed. ☐

P3.1 $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$. The base case is immediate from the definitions. Assume for induction that $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$. We know by **L3.4** that $|A \wedge B| = |A| \wedge |B|$ and by **L3.5** that $|A \vee B| = |A| \vee |B|$, where both $|A| \wedge |B|, |A| \vee |B| \in \mathbb{P}_{\mathcal{S}}$ by **L3.2** and **L3.3**. It follows that $|A \vee B|, |A \wedge B| \in \mathbb{P}_{\mathcal{S}}$. By induction, $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \texttt{pfs}(\mathcal{L}^-)$. ☐

L3.12 $\nvDash_{\mathcal{C}} (A \vee B) \wedge (A \vee C) \trianglelefteq A \vee (B \wedge C)$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. Since $x \cup y \in S$ for all $x, y \in S$, we may conclude that $\mathcal{S}_3$ is a state space. Additionally, both $\varnothing \cup s = s$ for all $s \in S$, and $S \cup s = S$ for all $s \in S$, and so $\mathcal{S}_3$ satisfies *Null State* and *Full State*. Observe that $x \cup x = x$, $x \cup y = y \cup x$, and $x \cup (y \cup z) = (x \cup y) \cup z$ for arbitrary $x, y, z \in S$, and so $\mathcal{S}_3$ also satisfies *Idempotency*, *Commutativity*, and *Associativity*. Thus $\mathcal{S}_3 \in \mathbb{M}$.

Let $\mathcal{M}_3$ be an $\mathcal{S}_3$-model where $|p_1|_3 = \{\{a\}\}$, $|p_2|_3 = \{\{b\}\}$, and $|p_3|_3 = \{\{c\}\}$, and all other assignments are arbitrary singletons. Given that $\mathcal{S}_3$ satisfies *Idempotency*, it follows immediately that $|p_i|_3 = \overline{|p_i|_3}$ for all $i \in \mathbb{N}$, and so $|p_i|_3 \in \mathbb{P}_{\mathcal{S}_3}$ for all $i \in \mathbb{N}$. Thus $\mathcal{M}_3 \in \mathcal{C}$.

It follows that $|p_1 \vee p_2|_3 = |p_1|_3 \vee |p_2|_3$ and $|p_1 \vee p_3|_3 = |p_1|_3 \vee |p_3|_3$ by **L3.5**, and so it follows by **L3.4** both $|p_2 \wedge p_3|_3 = |p_2|_3 \wedge |p_3|_3$ and $|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_3 = (|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3)$. Again by **L3.5**, we know that $|p_1 \vee (p_2 \wedge p_3)|_3 = |p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3)$.

By definition of $\vee$ and $\wedge$, we know that $|p_1|_3 \vee |p_2|_3 = \{\{a\}, \{b\}, \{a, b\}\}$, $|p_1|_3 \vee |p_3|_3 = \{\{a\}, \{c\}, \{a, c\}\}$, and $|p_2|_3 \wedge |p_3|_3 = \{\{b, c\}\}$, and so we know $(|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3) = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $|p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3) = \{\{a\}, \{b, c\}, \{a, b, c\}\}$. We may then observe that $(|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3) \nsubseteq |p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3)$. Given the **Unilateral Semantics**, $\mathcal{M}_3 \not\models (p_1 \vee p_2) \wedge (p_1 \vee p_3) \trianglelefteq p_1 \vee (p_2 \wedge p_3)$, and so $\not\models_{\mathcal{C}} (A \vee B) \wedge (A \vee C) \trianglelefteq A \vee (B \wedge C)$. ☐

L3.13 $\not\models_{\mathcal{C}} A \wedge (A \vee B) \trianglelefteq A$.

Proof. Let $\mathcal{M}_3$ be as in **L3.12**. Since $|p_1 \vee p_2|_3 = |p_1|_3 \vee |p_2|_3$ by **L3.5**, we know that $|p_1 \wedge (p_1 \vee p_2)|_3 = |p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3)$ by **L3.4**. However, $|p_1|_3 \vee |p_2|_3 = \{\{a\}, \{b\}, \{a, b\}\}$ where $|p_1|_3 = \{\{a\}\}$, and so it follows that $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) = \{\{a\}, \{a, b\}\}$. Since $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) \nsubseteq |p_1|_3$, we know that $\mathcal{M}_3 \not\models p_1 \wedge (p_1 \vee p_2) \trianglelefteq p_1$ by the **Unilateral Semantics**, and so may conclude that $\not\models_{\mathcal{C}} A \wedge (A \vee B) \trianglelefteq A$ as desired. ☐

L3.14 $\not\models_{\mathcal{C}} A \vee (A \wedge B) \trianglelefteq A$.

Proof. Let $\mathcal{M}_3$ be defined as in **L3.12**. Thus $|p_1 \wedge p_2|_3 = |p_1|_3 \wedge |p_2|_3$ by **L3.4**, and so $|p_1 \vee (p_1 \wedge p_2)|_3 = |p_1|_3 \vee (|p_1|_3 \wedge |p_2|_3)$ by **L3.5**. However, $|p_1|_3 \wedge |p_2|_3 = \{\{a, b\}\}$, and since $|p_1|_3 = \{\{a\}\}$, we may conclude that $|p_1|_3 \vee (|p_1|_3 \wedge |p_2|_3) = \{\{a\}, \{a, b\}\}$. Thus $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) \nsubseteq |p_1|_3$, and so we know that $\mathcal{M}_3 \not\models p_1 \vee (p_1 \wedge p_2) \trianglelefteq p_1$ by the **Unilateral Semantics**. Thus it follows that $\not\models_{\mathcal{C}} A \vee (A \wedge B) \trianglelefteq A$ as desired. ☐

L3.15 $\not\models_{\mathcal{C}} A \wedge (A \vee B) \approx A$.

Proof. Follows from **L3.13**. ☐

L3.16 $\not\models_{\mathcal{C}} A \vee (A \wedge B) \approx A$

Proof. Follows from **L3.14**. ☐

L3.17 $\not\models_{\mathcal{C}} A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$.

Proof. Follows from **L3.12**. ☐

L3.18 $X \wedge (X \vee Y) \neq X$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.15**, $\mathcal{M} \not\models p_1 \wedge (p_1 \vee p_2) \approx p_1$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, and so $|p_1 \wedge (p_1 \vee p_2)| \neq |p_1|$ by the **Unilateral Semantics**. Thus it follows by **L3.4** and **L3.5** that $|p_1| \wedge (|p_1| \vee |p_2|) \neq |p_1|$. Since $p_1, p_2 \in \texttt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existentially generalising on $|p_1|, |p_2|$ and $\mathcal{S} \in \mathbb{M}$. ☐

L3.19 $X \vee (X \wedge Y) \neq X$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.16**, $\mathcal{M} \nvDash p_1 \vee (p_1 \wedge p_2) \approx p_1$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, and so $|p_1 \vee (p_1 \wedge p_2)| \neq |p_1|$ by the **Unilateral Semantics**. Thus it follows by **L3.4** and **L3.5** that $|p_1| \vee (|p_1| \wedge |p_2|) \neq |p_1|$. Since $p_1, p_2 \in \texttt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existentially generalising on $|p_1|, |p_2|$ and $\mathcal{S} \in \mathbb{M}$. □

L3.20 For all $\mathcal{S} \in \mathbb{M}$ and $X, Y, U, V \in \mathbb{P}_{\mathcal{S}}$, if $X \sqsubseteq Y$ and $U \sqsubseteq V$, then $X \wedge U \sqsubseteq Y \wedge V$.

Proof. Let $\mathcal{S} \in \mathbb{M}$ and $X, Y, U, V \in \mathbb{P}_{\mathcal{S}}$, and assume $X \sqsubseteq Y$ and $U \sqsubseteq V$. Choose some $s \in X \vee U$. By *Sum*, $s \in X \cup U \cup (X \wedge U)$. If $s \in X \cup U$, then $s \in Y \cup V$, and so $s \in Y \cup V \cup (X \wedge V)$. If instead $s \in X \wedge U$, then $s = x \star u$ for some $x \in X$ and $u \in U$. It follows that $x \in Y$ and $u \in V$, and so $s \in Y \wedge V$. Thus $s \in Y \cup V \cup (Y \wedge V)$ in either case, and so $s \in Y \vee V$ by *Sum*. We may then conclude that $X \vee U \sqsubseteq Y \vee V$. □

L3.21 $X \vee (Y \wedge Z) \neq (X \vee Y) \wedge (X \vee Z)$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.17** above, we know that there is some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ where $\mathcal{M} \nvDash p_1 \vee (p_2 \wedge p_3) \approx (p_1 \vee p_2) \wedge (p_1 \vee p_3)$, and so it follows from the **Unilateral Semantics** that $|p_1 \vee (p_2 \wedge p_3)| \neq |(p_1 \vee p_2) \wedge (p_1 \vee p_3)|$. Thus $|p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|)$ by **L3.4** and **L3.5**. Since $p_1, p_2, p_3 \in \texttt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2|, |p_3| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existential generalisation. □

L3.22 $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$. We may then reason as follows:

$$\begin{aligned} X \wedge (Y \vee Z) &=_1 X \wedge [Y \cup Z \cup (Y \wedge Z)] \\ &=_2 (X \wedge Y) \cup (X \wedge Z) \cup [X \wedge (Y \wedge Z)] \\ &=_3 (X \wedge Y) \cup (X \wedge Z) \cup [(X \wedge X) \wedge (Y \wedge Z)] \\ &=_4 (X \wedge Y) \cup (X \wedge Z) \cup [(X \wedge Y) \wedge (X \wedge Z)] \\ &=_5 (X \wedge Y) \vee (X \wedge Z). \end{aligned}$$

Whereas both (1) and (5) are given by *Sum*, (2) follows from **L3.1** and (3) holds by **L2.13**. We may then justify (4) by appeal to **L2.12** and **L2.13**. Thus $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$. □

4 Soundness

Given the semantics above, we may now prove that UGSN is sound over $\mathcal{C}$ by induction on the number of applications of the meta-rules as follows:

T1 (*Soundness*) If $\Sigma \vdash_{\text{UGSN}} \varphi$, then $\Sigma \vDash_{\mathcal{C}} \varphi$.

Proof. The proof goes by a routine induction, drawing on **L4.1** – **L4.18** in addition to standard validities for propositional logic. □

It follows from **L3.15** –**L3.17** and *Soundness* that **#Abs1**, **#Abs2**, and **#Dist** are not theorems of UGSN. It remains to prove **L4.1** – **L4.18**.

L4.1 $\vDash_{\mathcal{C}} A \trianglelefteq A \vee B$ and $\vDash_{\mathcal{C}} B \trianglelefteq A \vee B$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A, B \in \texttt{pfs}(\mathcal{L}^-)$. Thus $|A \vee B| = |A| \vee |B|$ by **L3.5**, where $|A| \vee |B| = |A| \cup |B| \cup |A \wedge B|$ by *Sum*. It follows that $|A| \subseteq |A \vee B|$ and $|B| \subseteq |A \vee B|$, and so both $\mathcal{M} \vDash A \trianglelefteq A \vee B$ and $\mathcal{M} \vDash B \trianglelefteq A \vee B$ by the semantics. □

L4.2 $\vDash_{\mathcal{C}} A \trianglelefteq A \wedge A$ and $\vDash_{\mathcal{C}} A \wedge A \trianglelefteq A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$. By **L3.4**, $|A \wedge A| = |A| \wedge |A|$, where $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so $|A| \wedge |A| = |A|$ by **L2.13**. Thus $|A \wedge A| = |A|$, and so $\mathcal{M} \vDash A \trianglelefteq A \wedge A$ and $\mathcal{M} \vDash A \wedge A \trianglelefteq A$. □

L4.3 $\vDash_{\mathcal{C}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$ and $\vDash_{\mathcal{C}} (A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$.

Proof. Letting $\mathcal{M} \in \mathcal{C}$ and $A, B, C \in \texttt{pfs}(\mathcal{L}^-)$, it follows by **P3.1** that $|A|, |B|, |C| \in \mathbb{P}_{\mathcal{S}}$. Thus we may argue as follows:

$$\begin{aligned} |A \wedge (B \wedge C)| &=_1 |A| \wedge (|B| \wedge |C|) \\ &=_2 (|A| \wedge |B|) \wedge |C| \\ &=_3 |(A \wedge B) \wedge C|. \end{aligned}$$

Whereas both (1) and (3) follow from **L3.4**, (2) is given by **L3.8**. Thus $|A \wedge (B \wedge C)| = |(A \wedge B) \wedge C|$, and so $\mathcal{M} \vDash A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$ and $\mathcal{M} \vDash (A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$. □

L4.4 $\vDash_{\mathcal{C}} A \wedge B \trianglelefteq B \wedge A$.

Proof. Letting $\mathcal{M} \in \mathcal{C}$ and $A, B \in \texttt{pfs}(\mathcal{L}^-)$, it follows by **P3.1** that $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$. Thus we may argue as follows:

$$\begin{aligned} |A \wedge B| &=_1 |A| \wedge |B| \\ &=_2 |B| \wedge |A| \\ &=_3 |B \wedge A|. \end{aligned}$$

Whereas both (1) and (3) follow from **L3.4**, (2) is given by **L3.7**. Thus $|A \wedge B| = |B \wedge A|$, and so $\mathcal{M} \vDash A \wedge B \trianglelefteq B \wedge A$. □

L4.5 $A \trianglelefteq B, C \trianglelefteq D \vDash_{\mathcal{C}} A \wedge C \trianglelefteq B \wedge D$.

Proof. Assume $\mathcal{M} \vDash A \trianglelefteq B$ and $\mathcal{M} \vDash C \trianglelefteq D$ for some $\mathcal{M} \in \mathcal{C}$ and $A, B, C, D \in \texttt{pfs}(\mathcal{L}^-)$. Thus $|A| \subseteq |B|$ and $|C| \subseteq |D|$, where we know by **P3.1** that $|A|, |B|, |C|, |D| \in \mathbb{P}_{\mathcal{S}}$. By **L3.20**, $|A| \wedge |C| \subseteq |B| \wedge |D|$, and so $|A \wedge C| \subseteq |B \wedge D|$ by **L3.4**. Thus $\mathcal{M} \vDash A \wedge C \trianglelefteq B \wedge D$. □

L4.6 $A \trianglelefteq B, B \trianglelefteq C \vDash_{\mathcal{C}} A \trianglelefteq C$.

Proof. Assume $\mathcal{M} \vDash A \trianglelefteq B$ and $\mathcal{M} \vDash B \trianglelefteq C$ for some $\mathcal{M} \in \mathcal{C}$ and $A, B, C \in \texttt{pfs}(\mathcal{L}^-)$. Thus $|A| \subseteq |B|$ and $|B| \subseteq |C|$, and so $|A| \subseteq |C|$. We may then conclude that $\mathcal{M} \vDash A \trianglelefteq B$. □

L4.7 $\vDash_{\mathcal{C}} A \trianglelefteq \mathcal{T}$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$. By **P3.1**, $|A| \in \mathbb{P}_\mathcal{S}$, and so $|A| \subseteq S$. Thus, $|A| \subseteq |\mathcal{T}|$ since $|\mathcal{T}| = S$, and so $\mathcal{M} \vDash A \trianglelefteq \mathcal{T}$. □

L4.8 $\vDash_\mathcal{C} \$\bot\!\!\!\!\bot$.

Proof. Immediate from the semantics. □

L4.9 $\vDash_\mathcal{C} \$\mathcal{V}$.

Proof. Immediate from the semantics. □

L4.10 $\vDash_\mathcal{C} \bot\!\!\!\!\bot \wedge A \trianglelefteq A$ and $\vDash_\mathcal{C} A \trianglelefteq \bot\!\!\!\!\bot \wedge A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$. We may then argue as follows:

$$\begin{aligned} s \in |\bot\!\!\!\!\bot \wedge A| \quad & \textit{iff} \quad s = \square \star y \text{ for some } y \in |A| \\ (*) \quad & \textit{iff} \quad s = y \text{ for some } y \in |A| \\ & \textit{iff} \quad s \in |A|. \end{aligned}$$

Given that $(*)$ holds by *Null State*, it follows that $|\bot\!\!\!\!\bot \wedge A| = |A|$, and so both $\mathcal{M} \vDash \bot\!\!\!\!\bot \wedge A \trianglelefteq A$ and $\mathcal{M} \vDash A \trianglelefteq \bot\!\!\!\!\bot \wedge A$. □

L4.11 $\vDash_\mathcal{C} \mathcal{V} \wedge A \trianglelefteq \mathcal{V}$ and $\vDash_\mathcal{C} \mathcal{V} \trianglelefteq \mathcal{V} \wedge A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$. We may then argue as follows:

$$\begin{aligned} s \in |\mathcal{V} \wedge A| \quad & \textit{iff} \quad s = \blacksquare \star y \text{ for some } y \in |A| \\ (*) \quad & \textit{iff} \quad s = \blacksquare \\ & \textit{iff} \quad s \in |\mathcal{V}|. \end{aligned}$$

Given that $(*)$ holds by *Full State*, it follows that $|\mathcal{V} \wedge A| = |\mathcal{V}|$, and so both $\mathcal{M} \vDash \mathcal{V} \wedge A \trianglelefteq \mathcal{V}$ and $\mathcal{M} \vDash \mathcal{V} \trianglelefteq \mathcal{V} \wedge A$. □

L4.12 $\$A \vDash_\mathcal{C} A \ntrianglelefteq \bot$.

Proof. Assume that $\mathcal{M} \vDash \$A$ for some $\mathcal{M} \in \mathcal{C}$. It follows that $|A| = \{s\}$ for some $s \in S$, where $|\bot| = \varnothing$, and so $|A| \nsubseteq |\bot|$. Thus $\mathcal{M} \nvDash A \trianglelefteq \bot$, and so $\mathcal{M} \vDash A \ntrianglelefteq \bot$. □

L4.13 $A \approx B \vDash_\mathcal{C} \$A \leftrightarrow \$B$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and assume for contraposition that $\mathcal{M} \vDash \$A$ and $\mathcal{M} \nvDash \$B$. It follows that $|A| = \{s\}$ for some $s \in S$ where $|B| \neq \{s\}$, and so $|A| \neq |B|$. Thus either $|A| \nsubseteq |B|$ or $|B| \nsubseteq |A|$, and so either $\mathcal{M} \nvDash A \trianglelefteq B$ or $\mathcal{M} \nvDash B \trianglelefteq B$. We may then conclude that $\mathcal{M} \nvDash A \approx B$, where the same holds if $\mathcal{M} \nvDash \$A$ and $\mathcal{M} \vDash \$B$. □

L4.14 $\$A, \$B \vDash_\mathcal{C} \$(A \wedge B)$.

Proof. Assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash \$B$ for some $\mathcal{M} \in \mathcal{C}$. It follows that $|A| = \{s\}$ and $|B| = \{t\}$ for some $s, t \in S$, where we know by **L3.4** that $|A \wedge B| = \{x \star y : x \in |A| \text{ and } y \in |B|\}$. Thus $|A \wedge B| = \{s \star t\}$, and so $\mathcal{M} \vDash \$(A \wedge B)$. □

L4.15 $\$A, B \trianglelefteq A \vDash_{\mathcal{C}} (A \trianglelefteq B) \vee (B \trianglelefteq \bot)$.

Proof. Let $\mathcal{M} \in \mathcal{C}$, and assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash B \trianglelefteq A$. It follows that $|A| = \{s\}$ for some $s \in S$, and $|B| \subseteq |A|$, and so $|B| \subseteq \{s\}$. Thus either $|B| = \{s\}$ or $|B| = \varnothing$, and so either $|A| \subseteq |B|$ or $|B| \subseteq |\bot|$. In either case, $\mathcal{M} \vDash (A \trianglelefteq B) \vee (B \trianglelefteq \bot)$. □

L4.16 $\$A, A \trianglelefteq C \vee D \vDash_{\mathcal{C}} (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$.

Proof. Let $\mathcal{M} \in \mathcal{C}$, and assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash A \trianglelefteq C \vee D$. Thus $|A| = \{s\}$ for some $s \in S$, and $|A| \subseteq |C \vee D|$, where we know by **L3.5** and *Sum* that $|C \vee D| = |C| \cup |D| \cup |C \wedge D|$, and so $s \in |C| \cup |D| \cup |C \wedge D|$. Given that $|A| = \{s\}$, either $|A| \subseteq |C|$ or $|A| \subseteq |D|$ or $|A| \subseteq |C \wedge D|$, and so $\mathcal{M} \vDash A \trianglelefteq C$ or $\mathcal{M} \vDash A \trianglelefteq D$ or $\mathcal{M} \vDash A \trianglelefteq C \wedge D$. Thus $\mathcal{M} \vDash (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$. □

L4.17 If $\Gamma \vDash_{\mathcal{C}} \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$ where $p \in \mathbb{L}$ does not occur in A, B or any $\gamma \in \Gamma$, then $\Gamma \vDash_{\mathcal{C}} A \trianglelefteq B$.

Proof. Let $p \in \mathbb{L}$ be an arbitrary sentence letter which does not occur in A, B or any $\gamma \in \Gamma$. Assume for contraposition that $\Gamma \nvDash_{\mathcal{C}} A \trianglelefteq B$. Thus there is some $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, but $\mathcal{M} \nvDash A \trianglelefteq B$. It follows that $|A| \nsubseteq |B|$, and so there is some $s \in |A|$ such that $s \notin |B|$. Let $\mathcal{M}_u$ differ at most from $\mathcal{M}$ by setting $|p|_u = \{s\}$. Given that p does not occur in A or B, we may conclude that $|A| = |A|_u$ and $|B| = |B|_u$, and so both $|p|_u \subseteq |A|_u$ but $|p|_u \nsubseteq |B|_u$ given the above. Thus $\mathcal{M}_u \vDash \$p$ where $\mathcal{M}_u \vDash p \trianglelefteq A$ but $\mathcal{M}_u \nvDash p \trianglelefteq B$, and so $\mathcal{M}_u \nvDash (p \trianglelefteq A) \to (p \trianglelefteq B)$ and $\mathcal{M}_u \nvDash \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$. Since p does not occur in any $\gamma \in \Gamma$, and $\mathcal{M}_u$ differs from $\mathcal{M}$ at most in p, it follows that $\mathcal{M}_u \vDash \gamma$ for all $\gamma \in \Gamma$. Thus $\Gamma \nvDash_{\mathcal{C}} \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$. □

L4.18 If $\Gamma, [(A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \to [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)] \vDash_{\mathcal{C}} \varphi$ for distinct $p, q \in \mathbb{L}$ which do not occur in φ, A, B or any $\gamma \in \Gamma$, then $\Gamma \vDash_{\mathcal{C}} \varphi$.

Proof. Choose some distinct $p, q \in \mathbb{L}$ which do not occur in φ, A, B or any $\gamma \in \Gamma$, and assume $\Gamma \nvDash_{\mathcal{C}} \varphi$ for contraposition. It follows that there is some $\mathcal{M} \in \mathcal{C}$ where $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$ but $\mathcal{M} \nvDash \varphi$. Of course, either: (a) $\mathcal{M} \vDash (A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$; or (b) $\mathcal{M} \nvDash (A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. Assume (b) to start. It follows that:

$$\mathcal{M} \vDash [(A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \to [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)].$$

Given that $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, the antecedent of the claim to be proven is false. Assume (a) instead: $\mathcal{M} \vDash (A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. Thus $\mathcal{M} \nvDash A \trianglelefteq \bot$ and $\mathcal{M} \vDash A \trianglelefteq B \wedge C$, and so $|A| \nsubseteq \varnothing$ and $|A| \subseteq |B \wedge C|$. It follows that there is some $a \in |A|$, where $a \in |B \wedge C|$, and so $a = b \star c$ for some $b \in |B|$ and $c \in |C|$. Let $\mathcal{M}_u$ differ at most from $\mathcal{M}$ by setting $|p|_u = \{b\}$ and $|q|_u = \{c\}$, and so $\mathcal{M}_u \vDash \$p$ and $\mathcal{M}_u \vDash \$q$. Since p and q do not occur in A, B, φ or any $\gamma \in \Gamma$, we know that $\mathcal{M}_u \nvDash \varphi$ and $\mathcal{M}_u \vDash \gamma$ for all $\gamma \in \Gamma$. Additionally, $|A| = |A|_u$, $|B| = |B|_u$, and $|C| = |C|_u$, and so $|p|_u \subseteq |B|_u$ and $|q|_u \subseteq |C|_u$. Thus $\mathcal{M}_u \vDash p \trianglelefteq B$ and $\mathcal{M}_u \vDash q \trianglelefteq C$. Given **L3.4**, we may argue as follows:

$$\begin{aligned} |p \wedge q|_u &= \{x \star y : x \in |p|_u \text{ and } y \in |q|_u\} \\ &= \{b \star c\} \\ &= \{a\} \end{aligned}$$

Since $a \in |A| = |A|_u$, we know $|p \wedge q|_u \subseteq |A|_u$, and so $\mathcal{M}_u \vDash p \wedge q \trianglelefteq A$. Thus $\mathcal{M}_u \vDash \$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)$, and so trivially:

$$\mathcal{M}_u \vDash [(A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)].$$

Given that $\mathcal{M}_u \nvDash \varphi$ where $\mathcal{M}_u \vDash \gamma$ for all $\gamma \in \Gamma$, it follows that the antecedent of the claim to be proven is false. Thus the antecedent is false whether $\mathcal{M} \vDash (A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$ or $\mathcal{M} \nvDash (A \ntrianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. The claim to be proven then follows by discharge and contraposition. □

5 Completeness

Let $\mathcal{L}^+$ be a language like $\mathcal{L}^-$ but with $\mathbb{L}^+ = \mathbb{L} \cup \mathbb{Q} \cup \mathbb{W}$ in place of $\mathbb{L}$ where $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ and $\mathbb{W} = \bigcup_{i \in \mathbb{N}} \{r_i, s_i\}$. Keeping the formation rules the same as before, let $\texttt{pfs}(\mathcal{L}^+)$ be the set of pfs recursively generated from $\mathbb{L}^+$ rather than $\mathbb{L}$, where $\texttt{wfs}(\mathcal{L}^+)$ is then generated from $\texttt{pfs}(\mathcal{L}^+)$ *via* $\texttt{atoms}(\mathcal{L}^+)$ as above. In defining $\vdash_{\text{UGSN}}$, we may then permit instances of the axioms and rules of inference included in UGSN to draw upon both $\texttt{pfs}(\mathcal{L}^+)$ and $\texttt{wfs}(\mathcal{L}^+)$. Whereas in §3 and §4 we were concerned to evaluate truth relative to models in $\mathcal{C}$, we must now extend consideration to all $\texttt{wfs}(\mathcal{L}^+)$. Letting $\mathcal{M} = \langle S, \star, |\cdot|\rangle$ be a *unilateral* $\mathcal{S}$*-model of* $\mathcal{L}^+$ just in case $\mathcal{S} \in \mathbb{M}$ where $\mathcal{S} = \langle S, \star\rangle$ and $|p| \in \mathbb{P}_\mathcal{S}$ for all $p \in \mathbb{L}^+$, we may take $\mathcal{C}^+_\mathcal{S}$ to be the class of all $\mathcal{S}$-models of $\mathcal{L}^+$, where $\mathcal{C}^+ = \bigcup\{\mathcal{C}^+_\mathcal{S} : \mathcal{S} \in \mathbb{M}\}$. We are now in a position to prove that UGSN is complete over $\mathcal{C}$ by first showing that UGSN is complete over $\mathcal{C}^+$.

T2 (*Completeness*) If $\Sigma \vDash_\mathcal{C} \varphi$, then $\Sigma \vdash_{\text{UGSN}} \varphi$.

Proof. The proof that UGSN is complete over $\mathcal{C}$ goes by contraposition. Assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$. In order to prove that $\Sigma \nvDash_\mathcal{C} \theta$, I will construct a Henkin model $\mathcal{M}_{\Gamma_{\Sigma,\theta}}$ where $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \nvDash \theta$ for a carefully chosen set of wfss $\Gamma_{\Sigma,\theta} \subseteq \texttt{wfs}(\mathcal{L}^+)$. With this aim in mind, consider the following definitions:

Maximality: Γ is *maximal iff* for all $\varphi \in \texttt{wfs}(\mathcal{L}^+)$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Consistency: Γ is *consistent iff* $\nvdash_{\text{UGSN}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n)$ for any $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$.

Saturated: Γ is *saturated iff* for all $A, B \in \texttt{pfs}(\mathcal{L}^-)$, there is some $q \in \mathbb{Q}$ not occurring in A or B where $[(q \trianglelefteq A) \rightarrow (q \trianglelefteq B)] \rightarrow (A \trianglelefteq B) \in \Gamma$.

Conjunctive: Γ is *conjunctive iff* whenever $A \ntrianglelefteq \bot, A \trianglelefteq B \wedge C \in \Gamma$, then there are some $X, Y \in \texttt{pfs}(\mathcal{L}^+)$ where $\$X, \$Y, X \trianglelefteq B, Y \trianglelefteq C, X \wedge Y \trianglelefteq A \in \Gamma$.

$\trianglelefteq$-*Consistent:* Γ is $\trianglelefteq$-*consistent iff* Γ is saturated, conjunctive, and consistent.

Given the definitions above and any $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ where $\Sigma \nvdash_{\text{UGSN}} \theta$, we may construct a maximal $\trianglelefteq$-consistent set $\Gamma_{\Sigma,\theta}$, proving the following:

L5.6: For all $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Gamma_{\Sigma,\theta} \subseteq \texttt{wfs}(\mathcal{L}^+)$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$.

Given any maximal $\trianglelefteq$-consistent set $\Gamma_{\Sigma,\theta}$, I will show how to construct a Henkin model $\mathcal{M}_{\Gamma_{\Sigma,\theta}}$. Letting $\mathcal{M}^{\mathfrak{R}}$ be the $\mathcal{L}^-$-reduct of $\mathcal{M} \in \mathcal{C}^+$, we may prove:

L5.10: If $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}^+$.

L5.11: If $\mathcal{M} \in \mathcal{C}^+$, then $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{R}} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$.

P5.1: If $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, then $\chi \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$ for all $\chi \in \texttt{wfs}(\mathcal{L}^-)$.

Given the assumption that $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, it follows by **L5.6** that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$. By **P5.1**, $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \sigma$ for all $\sigma \in \Sigma$ where $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \nvDash \theta$. Given that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}^+$ by **L5.10**, it follows from **L5.11** that $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \vDash \varphi$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$. Thus $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \vDash \sigma$ for all $\sigma \in \Sigma$ where $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \nvDash \theta$, and so $\Sigma \nvDash_{\mathcal{C}} \theta$. By discharge and contraposition, we may conclude that if $\Sigma \vDash_{\mathcal{C}} \theta$ then $\Sigma \vdash_{\text{UGSN}} \theta$. It remains to establish each of the results stated above. □

In what follows, I will prove a number of preliminary results, culminating in proofs of the lemmas and propositions cited above.

L5.1 If $\Gamma \vdash_{\text{UGSN}} \varphi$ and $\Gamma \vdash_{\text{UGSN}} \neg\varphi$, then Γ is inconsistent.

Proof. Standard. □

L5.2 If Γ is a maximal consistent subset of $\texttt{wfs}(\mathcal{L}^+)$, then for all $\varphi \in \texttt{wfs}(\mathcal{L}^+)$:

(a) $\varphi \in \Gamma$ if and only if $\neg\varphi \notin \Gamma$;
(b) $\Gamma \vdash_{\text{UGSN}} \varphi$ if and only if $\varphi \in \Gamma$;
(c) $\varphi \vee \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\psi \in \Gamma$;
(d) $\varphi \wedge \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ and $\psi \in \Gamma$.

Proof. Standard. □

L5.3 If $X \cup Y$ is inconsistent for $Y \neq \varnothing$, then $X \vdash_{\text{UGSN}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n)$ for some $\{\gamma_1, \ldots, \gamma_n\} \subseteq Y$.

Proof. Standard. □

Given any set of sentences $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ for which $\Sigma \nvdash_{\text{UGSN}} \theta$, we may construct a maximal consistent set $\Gamma_{\Sigma,\theta}$ where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$. The construction will proceed in three stages. First we define a saturated set $\Delta_{\Sigma,\theta}$ where $\Sigma \cup \{\neg\theta\} \subseteq \Delta_{\Sigma,\theta}$, showing that $\Delta_{\Sigma,\theta}$ is consistent. We then move to extend $\Delta_{\Sigma,\theta}$ to a conjunctive set $\Omega_{\Sigma,\theta}$, showing that $\Omega_{\Sigma,\theta}$ is also consistent. Lastly, we define a maximal consistent extension $\Gamma_{\Sigma,\theta}$ of $\Delta_{\Sigma,\theta}$ in the usual manner. More specifically, consider the following definitions:

α-Ordering: Let $\texttt{atoms}(\mathcal{L}^+) = \{\alpha_i : i \in \mathbb{N}\}$.

α-Witnesses: $\delta_i = [(q_i^\star \trianglelefteq A) \to (q_i^\star \trianglelefteq B)] \to (A \trianglelefteq B)$ where $q_i^\star$ is the lowest indexed member of $\mathbb{Q}$ not occurring in A, B, or δ_j for any $j < i$.

Saturation: Let $\Delta^0_{\Sigma,\theta} = \Sigma \cup \{\neg\theta\}$, $\Delta^{n+1}_{\Sigma,\theta} = \Delta^n_{\Sigma,\theta} \cup \{\delta_{n+1}\} \cup \{\$q^\star_{n+1}\}$, and $\Delta_{\Sigma,\theta} = \bigcup_{n\in\mathbb{N}} \Delta^n_{\Sigma,\theta}$.

Given these definitions, we may move to establish the first consistency proof.

L5.4 For all $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Delta_{\Sigma,\theta}$ is consistent.

Proof. The proof goes by induction, showing that $\Delta^n_{\Sigma,\theta}$ is consistent for all $n \in \mathbb{N}$. Let $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ and assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$. Assume for *reductio* that $\Delta^0_{\Sigma,\theta}$ is inconsistent. It follows that $\Sigma \vdash_{\text{UGSN}} \neg\theta$ by **L5.3**, contradicting the above. Thus $\Delta^0_{\Sigma,\theta}$ is consistent by *reductio*.

Assume for induction that $\Delta^n_{\Sigma,\theta}$ is consistent. Assume for *reductio* that $\Delta^{n+1}_{\Sigma,\theta}$ is inconsistent. Since $\Delta^{n+1}_{\Sigma,\theta} = \Delta^n_{\Sigma,\theta} \cup \{\delta_{n+1}\} \cup \{\$q^\star_{n+1}\}$, it follows by **L5.3** that $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg(\delta_{n+1} \wedge \$q^\star_{n+1})$. Consider the following:

$$
\begin{array}{rll}
\Delta^n_{\Sigma,\theta} & \vdash_{\text{UGSN}} & \neg(\delta_{n+1} \wedge \$q^\star_{n+1}) \\
& \vdash_{\text{UGSN}} & \neg[([(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \to [A \trianglelefteq B]) \wedge \$q^\star_{n+1}] \\
& \vdash_{\text{UGSN}} & \neg\$q^\star_{n+1} \vee ([(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \wedge [A \ntrianglelefteq B]) \\
& \vdash_{\text{UGSN}} & (\$q^\star_{n+1} \to [(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)]) \wedge (\$q^\star_{n+1} \to [A \ntrianglelefteq B]) \\
& \vdash_{\text{UGSN}} & \$q^\star_{n+1} \to [A \ntrianglelefteq B] \\
& \vdash_{\text{UGSN}} & \$q^\star_{n+1} \to [(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \\
(*) & \vdash_{\text{UGSN}} & A \trianglelefteq B \\
& \vdash_{\text{UGSN}} & \neg\$q^\star_{n+1}
\end{array}
$$

The above follows by propositional logic with the exception of $(*)$ which follows by the **SP6**. Thus we may conclude that $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\$q^\star_{n+1}$ where $q^\star_{n+1}$ is the lowest indexed member of $\mathbb{Q}$ not occurring in A, B, or δ_m for any $m < n+1$. It follows by **AR2** that $\Delta^n_{\Sigma,\theta[q^\star_n/q^\star_{n+1}]} \vdash_{\text{UGSN}} \neg\$q^\star_n$, and so $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\$q^\star_n$ since $q^\star_{n+1}$ does not occur in $\Delta^n_{\Sigma,\theta}$. However, by construction $\$q^\star_n \in \Delta^n_{\Sigma,\theta}$, and so $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \$q^\star_n$. Thus $\Delta^n_{\Sigma,\theta}$ is inconsistent by **L5.1**, contradicting the above. By *reductio*, $\Delta^{n+1}_{\Sigma,\theta}$ is consistent, and so $\Delta_{\Sigma,\theta}$ is consistent by induction. □

Having proven that $\Delta_{\Sigma,\theta}$ is consistent for any non-theorem θ, we may now proceed to construct $\Omega_{\Sigma,\theta}$ from $\Delta_{\Sigma,\theta}$ as follows:

∧-Atoms: Let $\texttt{atoms}^\wedge(\mathcal{L}^+) = \{A \trianglelefteq B \wedge C : A, B, C \in \texttt{pfs}(\mathcal{L}^+)\}$.

β-Ordering: Let $\texttt{atoms}^\wedge(\mathcal{L}^+) = \{\beta_i : i \in \mathbb{N}\}$.

β-Witnesses: $\omega_i = [\$A \wedge (A \trianglelefteq B \wedge C)] \to [\$r^\star_i \wedge \$s^\star_i \wedge (r^\star_i \trianglelefteq B) \wedge (s^\star_i \trianglelefteq C) \wedge (r^\star_i \wedge s^\star_i \approx A)]$ where $r^\star_i$ and $s^\star_i$ are the lowest indexed member of $\mathbb{W}$ not occurring in A, B, C, or ω_j for any $j < i$.

Completion: Let $\Omega^0_{\Sigma,\theta} = \Delta_{\Sigma,\theta}$, $\Omega^{n+1}_{\Sigma,\theta} = \Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\}$, and $\Omega_{\Sigma,\theta} = \bigcup_{n\in\mathbb{N}} \Omega^n_{\Sigma,\theta}$.

We may now move to show that $\Omega_{\Sigma,\theta}$ is also consistent.

L5.5 For all $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Omega_{\Sigma,\theta}$ is consistent.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ and assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$. It follows by **L5.4** that $\Delta_{\Sigma,\theta}$ is consistent, establishing the base case.

Assume for induction that $\Omega^n_{\Sigma,\theta}$ is consistent. Assume for *reductio* that $\Omega^{n+1}_{\Sigma,\theta}$ is inconsistent. Since $\Omega^{n+1}_{\Sigma,\theta} = \Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\}$, it follows by **L5.3** that $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\omega_{n+1}$, and so $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\theta \to \neg\omega_{n+1}$. By contraposition, $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \omega_{n+1} \to \theta$, and so $\Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\} \vdash_{\text{UGSN}} \theta$. Since $\theta \in \texttt{wfs}(\mathcal{L}^-)$, neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ occur in θ. Additionally, $\mathbb{W} \cap \Delta_{\Sigma,\theta} = \varnothing$, and so neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ are in $\Omega^0_{\Sigma,\theta}$. Since by construction $r^\star_{n+1}$ and $s^\star_{n+1}$ are the lowest indexed member of $\mathbb{W}$ not occurring in ω_j for any $j < i$, we may conclude that neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ are in $\Omega^n_{\Sigma,\theta}$. It follows from $\Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\} \vdash_{\text{UGSN}} \theta$ that $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \theta$ by **SP7**. However, we also know that $\neg\theta \in \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta}$, and so $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\theta$. Thus by **L5.1**, $\Omega^n_{\Sigma,\theta}$ is inconsistent, contradicting the above. By *reductio*, $\Omega^{n+1}_{\Sigma,\theta}$ is consistent, and so $\Omega_{\Sigma,\theta}$ is consistent by induction. □

It remains to identify a maximal consistent extension of $\Omega_{\Sigma,\theta}$. In particular, consider the following Henkin construction:

φ-Ordering: Let $\texttt{wfs}(\mathcal{L}^+) = \{\varphi_n : n \in \mathbb{N}\}$.

$$\Gamma^0_{\Sigma,\theta} = \Omega_{\Sigma,\theta}$$

$$\Gamma^{n+1}_{\Sigma,\theta} = \begin{cases} \Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\} & \text{if } \Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\} \text{ is consistent} \\ \Gamma^n_{\Sigma,\theta} \cup \{\neg\varphi_{n+1}\} & \text{otherwise.} \end{cases}$$

$$\Gamma_{\Sigma,\theta} = \bigcup_{n\in\mathbb{N}} \Gamma^n_{\Sigma,\theta}.$$

We now move to show that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent and includes $\Sigma \cup \{\neg\theta\}$.

L5.6 For all $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ and assume $\Sigma \nvdash_{\text{UGSN}} \theta$ for discharge. By **L5.5**, $\Omega_{\Sigma,\theta}$ is consistent, and so $\Gamma^0_{\Sigma,\theta}$ is consistent. Assume for induction that $\Gamma^n_{\Sigma,\theta}$ is consistent. If $\Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\}$ is consistent, then $\Gamma^{n+1}_{\Sigma,\theta}$ is consistent. Assume instead that $\Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\}$ is inconsistent. Thus $\Gamma^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\varphi_{n+1}$ follows by **L5.3**, where $\Gamma^{n+1}_{\Sigma,\theta} = \Gamma_{\Sigma,\theta} \cup \{\neg\varphi_{n+1}\}$. Assume for *reductio* that $\Gamma^{n+1}_{\Sigma,\theta}$ is inconsistent. Again by **L5.3** it follows that $\Gamma^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\neg\varphi_{n+1}$, and so $\Gamma^n_{\Sigma,\theta}$ is inconsistent by **L5.1**. By *reductio*, $\Gamma^{n+1}_{\Sigma,\theta}$ is consistent. Thus $\Gamma_{\Sigma,\theta}$ is consistent by induction.

In order to show that $\Gamma_{\Sigma,\theta}$ is maximal, let $\varphi \in \texttt{wfs}(\mathcal{L}^+)$ be arbitrary. It follows that $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. Thus either $\Gamma^n_{\Sigma,\theta} = \Gamma^{n-1}_{\Sigma,\theta} \cup \{\varphi_n\}$ or $\Gamma^n_{\Sigma,\theta} = \Gamma^{n-1}_{\Sigma,\theta} \cup \{\neg\varphi_n\}$, and so either $\varphi \in \Gamma^n_{\Sigma,\theta}$ or $\neg\varphi \in \Gamma^n_{\Sigma,\theta}$. Since $\Gamma^n_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$ where φ was arbitrary, $\Gamma_{\Sigma,\theta}$ is maximal.

We next show that $\Gamma_{\Sigma,\theta}$ is saturated. Let $A, B \in \texttt{pfs}(\mathcal{L}^-)$. It follows that $A \trianglelefteq B = \alpha$ for some $\alpha \in \texttt{atoms}(\mathcal{L}^+)$. By construction, there is some $\delta \in \Delta_{\Sigma,\theta}$ where $\delta = [(q \trianglelefteq A) \rightarrow (q \trianglelefteq B)] \rightarrow (A \trianglelefteq B)$ and q is the lowest indexed member of $\mathbb{Q}$ not occurring in A or B. Since $\Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$ and $A, B \in \texttt{pfs}(\mathcal{L}^-)$ were arbitrary, it follows that $\Gamma_{\Sigma,\theta}$ is saturated.

Lastly, we show that $\Gamma_{\Sigma,\theta}$ is conjunctive. Assume for discharge that $\$A, A \trianglelefteq B \wedge C \in \Gamma_{\Sigma,\theta}$. It follows that $\$A \wedge (A \trianglelefteq B \wedge C) \in \Gamma_{\Sigma,\theta}$ by **L5.2d**, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$A \wedge (A \trianglelefteq B \wedge C)$. We also know that $A \trianglelefteq B \wedge C = \beta_i$ for some $\beta_i \in \texttt{atoms}^\wedge(\mathcal{L}^+)$. By construction, $\omega_i \in \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$, where:

$$\omega_i = [\$A \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$r^\star \wedge \$s^\star \wedge (r^\star \trianglelefteq B) \wedge (s^\star \trianglelefteq C) \wedge (r^\star \wedge s^\star \trianglelefteq A)].$$

Thus it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \omega_i$. Given the above, we may then conclude that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$r^\star \wedge \$s^\star \wedge (r^\star \trianglelefteq B) \wedge (s^\star \trianglelefteq C) \wedge (r^\star \wedge s^\star \trianglelefteq A)$. By existentially generalising on $r^\star$ and $s^\star$, there are some $X, Y \in \texttt{pfs}(\mathcal{L}^+)$ where $\Gamma \vdash_{\text{UGSN}} \$X$, $\Gamma \vdash_{\text{UGSN}} \$Y$, $\Gamma \vdash_{\text{UGSN}} X \trianglelefteq B$, $\Gamma \vdash_{\text{UGSN}} Y \trianglelefteq C$, as well as $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq A$. By **L5.2b**, there are some $X, Y \in \texttt{pfs}(\mathcal{L}^+)$ where $\$X$, $\$Y$, $X \trianglelefteq B$, $Y \trianglelefteq C$, $X \wedge Y \trianglelefteq A \in \Gamma$. Thus $\Gamma_{\Sigma,\theta}$ is conjunctive.

Having established that $\Gamma_{\Sigma,\theta}$ is maximal, consistent, saturated, and conjunctive, we may conclude that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent as needed. Moreover, $\Sigma \cup \{\neg\theta\} \subseteq \Gamma_{\Sigma,\theta}$ since $\Sigma \cup \{\neg\theta\} \subseteq \Delta^0_{\Sigma,\theta} \subseteq \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$. Thus by **L5.2a**, $\theta \notin \Gamma_{\Sigma,\theta}$, and so we may conclude by discharge. □

We are now in a position to construct a Henkin model $\mathcal{M}_\Gamma \in \mathcal{C}^+$, where Γ is any maximal $\trianglelefteq$-consistent subset of $\texttt{wfs}(\mathcal{L}^+)$. Consider the following:

Γ-*Class:* $[A]_\Gamma = \{X : A \approx X \in \Gamma\}$.

Γ-*States:* $S_\Gamma = \{[A]_\Gamma : \$A \in \Gamma\}$.

Γ-*Fusion:* $[A]_\Gamma \star [B]_\Gamma = [A \wedge B]_\Gamma$.

Γ-*Valuation:* $|l|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq l \in \Gamma\}$ for all $l \in \mathbb{L}^+$.

Γ-*Model:* $\mathcal{M}_\Gamma = \langle S_\Gamma, \star, |\cdot|_\Gamma\rangle$.

Given this construction, I will show that $\mathcal{M}_\Gamma$ is indeed a $\mathcal{S}$-model of $\mathcal{L}^+$, beginning with a few preliminary results.

L5.7 If Γ is maximal $\trianglelefteq$-consistent and $A, B \in \texttt{pfs}(\mathcal{L}^+)$, then $[A]_\Gamma = [B]_\Gamma$ iff $A \approx B \in \Gamma$.

Proof. Let $A, B \in \texttt{pfs}(\mathcal{L}^+)$, and assume $[A]_\Gamma = [B]_\Gamma$. By **E1** we know that $\vdash_{\text{UGSN}} B \approx B$, and so $B \approx B \in \Gamma$ by **L5.2b**. Thus $B \in [B]_\Gamma$. Given the identity above, $B \in [A]_\Gamma$, and so $A \approx B \in \Gamma$ by definition.

Assume instead that $A \approx B \in \Gamma$. We may then argue as follows:

$$\begin{aligned} X \in [A]_\Gamma \quad & \textit{iff} \;\; A \approx X \in \Gamma \\ & \textit{iff} \;\; \Gamma \vdash_{\text{UGSN}} A \approx X \\ (*) \quad & \textit{iff} \;\; \Gamma \vdash_{\text{UGSN}} B \approx X \\ & \textit{iff} \;\; B \approx X \in \Gamma \\ & \textit{iff} \;\; X \in [B]_\Gamma. \end{aligned}$$

The identities above all hold by definition and **L5.2b**, with the exception of (∗) which follows from **GA9**. Thus $[A]_\Gamma = [B]_\Gamma$. Given the above, we may conclude that $[A]_\Gamma = [B]_\Gamma$ just in case $A \approx B \in \Gamma$. □

L5.8 If Γ is maximal $\trianglelefteq$-consistent, then $x \star y \in S_\Gamma$ for all $x, y \in S_\Gamma$.

Proof. Assume that Γ is maximal $\trianglelefteq$-consistent. In order to show that $\star$ is well-defined over S_Γ, let $a_1, a_2, b_1, b_2 \in S_\Gamma$, and assume that both $a_1 = a_2$ and $b_1 = b_2$. By definition, $a_1 = [A_1]_\Gamma$, $a_2 = [A_2]_\Gamma$, $b_1 = [B_1]_\Gamma$, and $b_2 = [B_2]_\Gamma$ for some $A_1, A_2, B_1, B_2 \in \texttt{pfs}(\mathcal{L}^+)$. Thus it follows that $[A_1]_\Gamma = [A_2]_\Gamma$ and $[B_1]_\Gamma = [B_2]_\Gamma$, and so $A_1 \approx A_2 \in \Gamma$ and $B_1 \approx B_2 \in \Gamma$ by **L5.7**. By **GA8**, we know that $\Gamma \vdash_{\text{UGSN}} A_1 \wedge B_1 \approx A_2 \wedge B_2$, and so $A_1 \wedge B_1 \approx A_2 \wedge B_2 \in \Gamma$ by **L5.2b**. Again by **L5.7**, we know that $[A_1 \wedge B_1]_\Gamma = [A_2 \wedge B_2]_\Gamma$, and so $[A_1]_\Gamma \star [B_1]_\Gamma = [A_2]_\Gamma \star [B_2]_\Gamma$ by definition. Thus $a_1 \star b_1 = a_2 \star b_2$, and so $\star$ is well-defined.

Let $x, y \in S_\Gamma$ by arbitrary. It follows that $x = [A]_\Gamma$ where $\$A \in \Gamma$, and similarly $y = [B]_\Gamma$ where $\$B \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} \$(A \wedge B)$ follows from **SP3**, and so $\$(A \wedge B) \in \Gamma$ by **L5.2b**. We may then conclude that $[A \wedge B]_\Gamma \in S_\Gamma$. However, $[A \wedge B]_\Gamma = [A]_\Gamma \star [B]_\Gamma$, and so $x \star y \in S_\Gamma$. Thus we may conclude that $x \star y \in S_\Gamma$ for all $x, y \in S_\Gamma$. □

L5.9 If Γ is maximal $\trianglelefteq$-consistent, then $\mathcal{M}_\Gamma \in \mathcal{C}^+$.

Proof. Assume Γ is maximal $\trianglelefteq$-consistent. It follows by **L5.8** that $\langle S_\Gamma, \star\rangle$ is a state space. We must show that $\langle S_\Gamma, \star\rangle \in \mathbb{M}$ by satisfying *Null State*, *Full State*, *Idempotency*, *Commutativity*, and *Associativity*.

Observe that $\vdash_{\text{UGSN}} \$\bot$ and $\vdash_{\text{UGSN}} \$\mathcal{V}$ follow from **VF1** and **VF2**, respectively. Thus $\Gamma \vdash_{\text{UGSN}} \$\bot$ and $\Gamma \vdash_{\text{UGSN}} \$\mathcal{V}$, and so $\$\bot, \$\mathcal{V} \in \Gamma$ by **L5.2b**. By Γ-*States*, both $[\bot]_\Gamma, [\mathcal{V}]_\Gamma \in S_\Gamma$. Letting $x \in S_\Gamma$, it follows by Γ-*States* that $x = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$, and so:

$$\begin{aligned} [\bot]_\Gamma \star x &= [\bot]_\Gamma \star [A]_\Gamma \\ &= [\bot \wedge A]_\Gamma \\ (\dagger) \quad &= [A]_\Gamma \\ &= x. \end{aligned} \qquad \begin{aligned} [\mathcal{V}]_\Gamma \star x &= [\mathcal{V}]_\Gamma \star [A]_\Gamma \\ &= [\mathcal{V} \wedge A]_\Gamma \\ (\ddagger) \quad &= [\mathcal{V}]_\Gamma. \end{aligned}$$

The identities above hold by definition or assumption with the exception of (†) and (‡) which follow by **L5.7** from **E4** and **E6**, respectively. Since $x \in S_\Gamma$ was arbitrary, we may conclude that there is some $\square \in S_\Gamma$ where $\square \star x = x$ for all $x \in S_\Gamma$, and some $\blacksquare \in S_\Gamma$ where $\blacksquare \star x = \blacksquare$ for all $x \in S_\Gamma$. Thus $\langle S_\Gamma, \star\rangle$ satisfies both *Null State* and *Full State*.

Letting $x \in S_\Gamma$. It follows that $x = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge A \approx A$ by **E4**, and so $A \wedge A \approx A \in \Gamma$ by **L5.2b**. Thus $[A \wedge A]_\Gamma = [A]_\Gamma$ by **L5.7**, and so $[A]_\Gamma \star [A]_\Gamma = [A]_\Gamma$. Equivalently, $x \star x = x$. Since $x \in S_\Gamma$ was arbitrary, $\langle S_\Gamma, \star\rangle$ satisfies *Idempotency*.

Let $x, y \in S_\Gamma$. It follows that $x = [A]_\Gamma$ and $y = [B]_\Gamma$ for some $A, B \in \texttt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge B \approx B \wedge A$ by **E10**, and so $A \wedge B \approx B \wedge A \in \Gamma$ by **L5.2b**. Thus $[A \wedge B]_\Gamma = [B \wedge A]_\Gamma$ by **L5.7**. By definition, $[A]_\Gamma \star [B]_\Gamma = [B]_\Gamma \star [A]_\Gamma$, and so $x \star y = y \star x$. Since $x, y \in S_\Gamma$ were arbitrary, it follows that $\langle S_\Gamma, \star\rangle$ satisfies *Commutativity*.

Let $x, y, z \in S_\Gamma$. Thus $x = [A]_\Gamma$, $y = [B]_\Gamma$, and $z = [C]_\Gamma$ for some $A, B, C \in \texttt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$ by **E12**, and so $A \wedge (B \wedge C) \approx (A \wedge B) \wedge C \in \Gamma$ by **L5.2b**. Thus $[A \wedge (B \wedge C)]_\Gamma = [(A \wedge B) \wedge C]_\Gamma$ by **L5.7**. Consider the following:

$$\begin{aligned} x \star (y \star z) &= [A]_\Gamma \star ([B]_\Gamma \star [C]_\Gamma) \\ &= [A \wedge (B \wedge C)]_\Gamma \\ (*) &= [(A \wedge B) \wedge C]_\Gamma \\ &= ([A]_\Gamma \star [B]_\Gamma) \star [C]_\Gamma. \\ &= (x \star y) \star z. \end{aligned}$$

The identities above all follow by definition or assumption with the exception of $(*)$ which was already established. Since $x, y, z \in S_\Gamma$ were arbitrary, $\langle S_\Gamma, \star\rangle$ satisfies *Associativity* as desired.

Given the results above, $\langle S_\Gamma, \star\rangle \in \mathbb{M}$. We now show that $|p|_\Gamma \in \mathbb{P}_{\mathcal{S}_\Gamma}$ for all $p \in \mathbb{L}^+$. Letting $p \in \mathbb{L}^+$, we know that $|p|_\Gamma \subseteq S_\Gamma$, and so $|p|_\Gamma \subseteq \overline{|p|_\Gamma}$. In order to establish the converse inclusion, let $s \in \overline{|p|_\Gamma}$. It follows that $s = x \star y$ for some $x, y \in |p|_\Gamma$, and so $x = [X]_\Gamma$ and $y = [Y]_\Gamma$ for some $X, Y \in \texttt{pfs}(\mathcal{L}^+)$ where both $\$X, \$Y \in \Gamma$ and $X \trianglelefteq p, Y \trianglelefteq p \in \Gamma$. By **SP3**, $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$, and so $\$(X \wedge Y) \in \Gamma$ by **L5.2b**. Since we also know that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq p \wedge p$ by **GA8**, it follows that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq p$ by **GA4**. Again by **L5.2b**, $X \wedge Y \trianglelefteq p \in \Gamma$, and so $[X \wedge Y]_\Gamma \in |p|_\Gamma$ by Γ-*Valuation*. By definition, $[X \wedge Y]_\Gamma = [X]_\Gamma \star [Y]_\Gamma$. It follows that $x \star y \in |p|_\Gamma$, and so $s \in |p|_\Gamma$. Since $s \in \overline{|p|_\Gamma}$ was arbitrary, $\overline{|p|_\Gamma} \subseteq |p|_\Gamma$. Given the above, $|p|_\Gamma = \overline{|p|_\Gamma}$, and so $|p|_\Gamma \in \mathbb{P}_{\mathcal{S}_\Gamma}$ where $\mathcal{S}_\Gamma = \langle S_\Gamma, \star\rangle$. Since $p \in \mathbb{L}^+$ was arbitrary and $\langle S_\Gamma, \star\rangle \in \mathbb{M}$, and so $\mathcal{M}_\Gamma \in \mathcal{C}^+$. □

L5.10 If Γ is maximal $\trianglelefteq$-consistent, then for all $B \in \texttt{pfs}(\mathcal{L}^+)$ $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$.

Proof. Assuming that Γ is maximal $\trianglelefteq$-consistent, the proof proceeds by induction on complexity. Assume to start that $B \in \texttt{pfs}(\mathcal{L}^+)$ where $\texttt{comp}(B) = 0$. It follows that either: (1) $B = \perp\!\!\!\!\perp$; (2) $B = \perp$; (3) $B = \mathcal{T}$; (4) $B = \mathcal{V}$; or (5) $B = p$ for some $p \in \mathbb{L}^+$. Since (5) is given immediately by Γ-*Valuation*, we may restrict consideration to (1) – (4).

Case 1: Assume $B = \perp\!\!\!\!\perp$. We know that $|\perp\!\!\!\!\perp|_\Gamma = \{\square\}$, where $\vdash_{\text{UGSN}} \$\perp\!\!\!\!\perp$ by **VF1** and $\vdash_{\text{UGSN}} \perp\!\!\!\!\perp \trianglelefteq \perp\!\!\!\!\perp$ by **E1**, and so both $\$\perp\!\!\!\!\perp \in \Gamma$ and $\perp\!\!\!\!\perp \trianglelefteq \perp\!\!\!\!\perp \in \Gamma$ by **L5.2b**. Since $\square = [\perp\!\!\!\!\perp]_\Gamma$, it follows that $\square \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma\}$. Let $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma\}$. By definition, $x = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$ where both $\$A \in \Gamma$ and $A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \perp$ and $\Gamma \vdash_{\text{UGSN}} (\perp\!\!\!\!\perp \trianglelefteq A) \vee (A \trianglelefteq \perp)$ follow by **SP1** and **SP4**, respectively. It follows by **L5.2** that $A \trianglelefteq \perp \notin \Gamma$ and either $\perp\!\!\!\!\perp \trianglelefteq A \in \Gamma$ or $A \trianglelefteq \perp \in \Gamma$, and so $\perp\!\!\!\!\perp \trianglelefteq A \in \Gamma$. Having already shown that $A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma$, it follows by **L5.2b** that $A \approx \perp\!\!\!\!\perp \in \Gamma$, and so $[A]_\Gamma = [\perp\!\!\!\!\perp]_\Gamma$ by **L5.7**. Thus $x = \square$, and so $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma\} = \{\square\}$ since x was arbitrary. Given the above, $|\perp\!\!\!\!\perp|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp\!\!\!\!\perp \in \Gamma\}$ as desired.

Case 2: Assuming $B = \perp$, it follows that $|\perp|_\Gamma = \varnothing$. Assume for *reducto* that there is some $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp \in \Gamma\}$. Thus $x = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$ and $A \trianglelefteq \perp \in \Gamma$. It follows that $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \perp$ by **SP1**, and so $A \trianglelefteq \perp \notin \Gamma$ by **L5.2**. Thus $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp \in \Gamma\} = \varnothing$ by *reductio*, and so we may conclude that $|\perp|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \perp \in \Gamma\}$.

Case 3: Assume $B = \mathcal{T}$. Thus $|\mathcal{T}|_\Gamma = S_\Gamma$. Letting $s \in S_\Gamma$, it follows that $s = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$. By **VF7**, $\vdash_{\text{UGSN}} A \trianglelefteq \mathcal{T}$, and so $A \trianglelefteq \mathcal{T} \in \Gamma$ by **L5.2b**. Thus $s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\}$. Since $s \in S_\Gamma$ was arbitrary, $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\} = S_\Gamma$. It follows that $|\mathcal{T}|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\}$ by the above.

Case 4: Assuming $B = \mathcal{V}$, it follows that $|\mathcal{V}|_\Gamma = \{\blacksquare\}$, where $\vdash_{\text{UGSN}} \$\mathcal{V}$ by **VF1** and $\vdash_{\text{UGSN}} \mathcal{V} \trianglelefteq \mathcal{V}$ by **E1**, and so both $\$\mathcal{V} \in \Gamma$ and $\mathcal{V} \trianglelefteq \mathcal{V} \in \Gamma$ by **L5.2b**. Since $\blacksquare = [\mathcal{V}]_\Gamma$, we know that $\blacksquare \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$. Let $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$. By definition, $x = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^+)$ where both $\$A \in \Gamma$ and $A \trianglelefteq \mathcal{V} \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \bot$ and $\Gamma \vdash_{\text{UGSN}} (\mathcal{V} \trianglelefteq A) \vee (A \trianglelefteq \bot)$ follow by **SP1** and **SP4**, respectively. It follows by **L5.2** that $A \trianglelefteq \bot \notin \Gamma$ and either $\mathcal{V} \trianglelefteq A \in \Gamma$ or $A \trianglelefteq \bot \in \Gamma$, and so $\mathcal{V} \trianglelefteq A \in \Gamma$. Having already shown that $A \trianglelefteq \mathcal{V} \in \Gamma$, it follows by **L5.2b** that $A \approx \mathcal{V} \in \Gamma$, and so $[A]_\Gamma = [\mathcal{V}]_\Gamma$ by **L5.7**. Thus $x = \blacksquare$, and so $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\} = \{\blacksquare\}$ since x was arbitrary. Given the above, $|\mathcal{V}|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$.

Having established the base cases, we may assume for induction that $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$ for all $B \in \texttt{pfs}(\mathcal{L}^+)$ such that $\texttt{comp}(B) < n$. Letting $B \in \texttt{pfs}(\mathcal{L}^+)$ be such that $\texttt{comp}(B) = n$, it follows that either $B = C \wedge D$ or $B = C \vee D$. Consider the following:

Case 1: Assume $B = C \wedge D$. Since both $\texttt{comp}(C), \texttt{comp}(D) < n$, we know by hypothesis that $|C|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\}$ and $|D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\}$. Consider the following:

$$
\begin{aligned}
s \in |C \wedge D|_\Gamma \quad & \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C \wedge D \\
& \textit{iff} \quad s = c \star d \text{ for some } c, d \text{ where } \mathcal{M}_\Gamma, c \Vdash C \text{ and } \mathcal{M}_\Gamma, d \Vdash D \\
& \textit{iff} \quad s = c \star d \text{ for some } c \in |C|_\Gamma \text{ and } d \in |D|_\Gamma \\
& \textit{iff} \quad s = c \star d \text{ for some } c \in \{[X]_\Gamma : \$X \in \Gamma \text{ and } X \trianglelefteq C \in \Gamma\} \\
& \qquad\qquad \text{and } d \in \{[Y]_\Gamma : \$Y \in \Gamma \text{ and } Y \trianglelefteq D \in \Gamma\} \\
& \textit{iff} \quad s = [X]_\Gamma \star [Y]_\Gamma \text{ for some } X, Y \in \texttt{pfs}(\mathcal{L}^+) \text{ such that } \$X, \$Y \in \Gamma \\
& \qquad\qquad \text{where both } X \trianglelefteq C, Y \trianglelefteq D \in \Gamma \\
(\dagger) \quad & \textit{iff} \quad s = [X \wedge Y]_\Gamma \text{ such that } \$(X \wedge Y) \in \Gamma \text{ where } X \wedge Y \trianglelefteq C \wedge D \in \Gamma \\
(\ddagger) \quad & \textit{iff} \quad s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}. \qquad [\wedge]
\end{aligned}
$$

The biconditionals above hold by definition or assumption with the exception of (†) and (‡). Starting with (†), assume that $s = [X]_\Gamma \star [Y]_\Gamma$ for some X and Y such that $\$X, \$Y \in \Gamma$ where $X \trianglelefteq C \in \Gamma$ and $Y \trianglelefteq D \in \Gamma$. It follows by **SP3** that $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$, and so $\$(X \wedge Y) \in \Gamma$ by **L5.2b**. We also know that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq C \wedge D$ by **GA8**, and so $X \wedge Y \trianglelefteq C \wedge D \in \Gamma$ by **L5.2b**. Additionally, it follows that $s = [X \wedge Y]_\Gamma$ by Γ-*Fusion*. Altogether, $s = [X \wedge Y]_\Gamma$ such that $\$(X \wedge Y) \in \Gamma$ where $X \wedge Y \trianglelefteq C \wedge D \in \Gamma$. Existentially, generalising on $X \wedge Y$, we may conclude that there is some $Z \in \texttt{pfs}(\mathcal{L}^+)$ such that $s = [Z]_\Gamma$ where $\$Z \in \Gamma$ and $Z \trianglelefteq C \wedge D \in \Gamma$, and so $s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}$.

Assume instead that $s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}$, and so for some $Z \in \texttt{pfs}(\mathcal{L}^+)$, $s = [Z]_\Gamma$ where $\$Z \in \Gamma$ and $Z \trianglelefteq C \wedge D \in \Gamma$. Since Γ is $\trianglelefteq$-consistent, we know Γ is conjunctive, and so there are some $X, Y \in \texttt{pfs}(\mathcal{L}^+)$ where $\$X, \$Y, X \trianglelefteq C, Y \trianglelefteq D, X \wedge Y \trianglelefteq Z \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq C \wedge D$ by **GA8**, and $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$ by **SP3**. We may then conclude that $\Gamma \vdash_{\text{UGSN}} (Z \trianglelefteq X \wedge Y) \vee (X \wedge Y \trianglelefteq \bot)$ by **SP4**, and so either $Z \trianglelefteq X \wedge Y \in \Gamma$ or $X \wedge Y \trianglelefteq \bot \in \Gamma$ by **L5.2**. We also know that $\$(X \wedge Y) \vdash_{\text{UGSN}} X \wedge Y \ntrianglelefteq \bot$ by **SP1**, and so $\Gamma \vdash_{\text{UGSN}} X \wedge Y \ntrianglelefteq \bot$. Thus $X \wedge Y \trianglelefteq \bot \notin \Gamma$ by **L5.2**, and so $Z \trianglelefteq X \wedge Y \in \Gamma$ given the above. Given that $X \wedge Y \trianglelefteq Z \in \Gamma$, it follows that $\Gamma \vdash_{\text{UGSN}} Z \approx X \wedge Y$, and so $Z \approx X \wedge Y \in \Gamma$ by **L5.2b**. Thus $[Z]_\Gamma = [X]_\Gamma \star [Y]_\Gamma$ by **L5.7**, and so $s = [X]_\Gamma \star [Y]_\Gamma$ for some X and Y where $\$X, \$Y \in \Gamma$ and $X \trianglelefteq C, Y \trianglelefteq D \in \Gamma$ since $s = [Z]_\Gamma$. Together with the forward direction, it follows that $|C \wedge D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \wedge D \in \Gamma\}$.

Case 2: Assume $B = C \vee D$. Since both $\texttt{comp}(C), \texttt{comp}(D) < n$, we know by hypothesis that $|C|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\}$ and $|D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\}$. Consider the following:

$$\begin{aligned}
s \in |C \vee D|_\Gamma \quad & \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C \vee D \\
& \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C, \text{ or } \mathcal{M}_\Gamma, s \Vdash D, \text{ or } \mathcal{M}_\Gamma, s \Vdash C \wedge D \\
& \textit{iff} \quad s \in |C|_\Gamma, \text{ or } s \in |D|_\Gamma, \text{ or } s \in |C \wedge D|_\Gamma \\
& \textit{iff} \quad s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\} \text{ or } s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\} \\
& \qquad\qquad \text{or } s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \wedge D \in \Gamma\} \\
& \textit{iff} \quad s = [A]_\Gamma \text{ for some } A \in \texttt{pfs}(\mathcal{L}^+) \text{ where } \$A \in \Gamma \text{ and} \\
& \qquad\qquad \text{and either } A \trianglelefteq C \in \Gamma \text{ or } A \trianglelefteq D \in \Gamma \text{ or } A \trianglelefteq C \wedge D \in \Gamma \\
(*) \quad & \textit{iff} \quad s = [A]_\Gamma \text{ for some } A \in \texttt{pfs}(\mathcal{L}^+) \text{ where } \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma \\
& \textit{iff} \quad s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma\}. \qquad [\vee]
\end{aligned}$$

All of the biconditionals above follow by assumption or definition, with the exception of $(*)$. For the forward direction, let $s = [A]_\Gamma$ for some $A \in \texttt{pfs}(\mathcal{L}^-)$ where $\$A \in \Gamma$ and either $A \trianglelefteq C \in \Gamma$ or $A \trianglelefteq D \in \Gamma$ or $A \trianglelefteq C \wedge D \in \Gamma$. Given **GA1**, **GA2**, and **T5**, we know by **L5.2b** that $C \trianglelefteq C \vee D \in \Gamma$, $D \trianglelefteq C \vee D \in \Gamma$ and $C \wedge D \trianglelefteq C \vee D \in \Gamma$. Thus it follows by **GA9** that $\Gamma \vdash_{\text{UGSN}} A \trianglelefteq C \vee D$ in each case, and so $A \trianglelefteq C \vee D \in \Gamma$ by **L5.2b**. Together with the above, $s = [A]_\Gamma$ for some A where $\$A \in \Gamma$ and $A \trianglelefteq C \vee D \in \Gamma$, thereby establishing the forward direction.

Assume instead that $s = [A]_\Gamma$ for some A where both $\$A \in \Gamma$ and $A \trianglelefteq C \vee D \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$ follows by **SP5**, and so either $A \trianglelefteq C \in \Gamma$, $A \trianglelefteq D \in \Gamma$, or $A \trianglelefteq C \wedge D \in \Gamma$ by **L5.2**. Thus it follows that $s = [A]_\Gamma$ for some A where $\$A \in \Gamma$ and either $A \trianglelefteq C \in \Gamma$, $A \trianglelefteq D \in \Gamma$, or $A \trianglelefteq C \wedge D \in \Gamma$. Together with the above, it follows that $|C \vee D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma\}$.

Given the cases proven above, it follows by induction on complexity that $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$ for all $B \in \texttt{pfs}(\mathcal{L}^+)$. □

It remains to show that every model $\mathcal{M} \in \mathcal{C}^+$ of the expanded language $\mathcal{L}^+$ has a *reduct* $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$ such that $\mathcal{M}^{\mathfrak{R}} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$. Given any $\mathcal{S}$-model $\mathcal{M} = \langle S, \star, |\cdot| \rangle \in \mathcal{C}^+$, consider the following:

Restriction: Let $|\cdot|^{\mathfrak{R}} : \mathbb{L} \to \mathbb{P}_{\mathcal{S}}$ where $|p|^{\mathfrak{R}} = |p|$ for all $p \in \mathbb{L}$.

$\mathfrak{R}$-*Map:* Let $\mathcal{M}^{\mathfrak{R}} = \langle S, \star, |\cdot|^{\mathfrak{R}} \rangle$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle \in \mathcal{C}^+$.

The following lemma proves that $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$ for any $\mathcal{M} \in \mathcal{C}^+$, where $\mathcal{M}^{\mathfrak{R}}$ makes the same wfs of $\mathcal{L}^-$ true as $\mathcal{M}$.

L5.11 If $\mathcal{M} \in \mathcal{C}^+$, then $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{R}} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$.

Proof. Let $\mathcal{M} \in \mathcal{C}^+$. By definition, $\mathcal{M} = \langle S, \star, |\cdot|\rangle$ where $\langle S, \star\rangle \in \mathbb{M}$ and $|p_i|, |q_i|, |r_i|, |s_i| \in \mathbb{P}_{\mathcal{S}}$ for all $i \in \mathbb{N}$. Thus $\mathcal{M}^{\mathfrak{R}} = \langle S, \star, |\cdot|\rangle$ where $\langle S, \star\rangle \in \mathbb{M}$ and $|p_i| \in \mathbb{P}_{\mathcal{S}}$ for all $i \in \mathbb{N}$, and so $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$.

By construction, $|p_i|^{\mathfrak{R}} = |p_i|$ for all $p_i \in \mathbb{L}$. Thus it follows by two routine induction proofs that: (1) $|A|^{\mathfrak{R}} = |A|$ for all $A \in \texttt{pfs}(\mathcal{L}^-)$; and (2) $\mathcal{M}^{\mathfrak{R}} \vDash \varphi$ just in case $\mathcal{M} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$ as desired. □

L5.12 $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$ for all $A, B \in \texttt{pfs}(\mathcal{L}^+)$.

Proof. Let $A, B \in \texttt{pfs}(\mathcal{L}^+)$, and assume $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$. Let $s \in |A|_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, $s = [X]_{\Gamma_{\Sigma,\theta}}$ where $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \trianglelefteq B$ and $X \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ by **GA9** and **L5.2b**, respectively. Given that $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $[X]_{\Gamma_{\Sigma,\theta}} \in |B|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**, and so $s \in |B|_{\Gamma_{\Sigma,\theta}}$. Thus $|A|_{\Gamma_{\Sigma,\theta}} \subseteq |B|_{\Gamma_{\Sigma,\theta}}$, and so $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$. It follows that if $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$.

Assume instead that $A \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$. It follows that $A \ntrianglelefteq B \in \Gamma_{\Sigma,\theta}$ and $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} A \ntrianglelefteq B$ by **L5.2a** and **L5.2b**, respectively. We also know that $A \trianglelefteq B \in \texttt{atoms}(\mathcal{L}^-)$, and so $A \trianglelefteq B = \alpha_i$ for some $i \in \mathbb{N}$. By construction, $\omega_i, \$q_i \in \Delta^i_{\Sigma,\theta}$ where $\omega_i = [(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)] \to (A \trianglelefteq B)$. Since $\Delta^i_{\Sigma,\theta} \subseteq \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$, it follows that $\omega_i, \$q_i \in \Gamma_{\Sigma,\theta}$. We may then conclude that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} [(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)] \to (A \trianglelefteq B)$, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg[(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)]$ since $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} A \ntrianglelefteq B$. Thus $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} (q_i \trianglelefteq A) \wedge (q_i \ntrianglelefteq B)$, and so $q_i \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ and $q_i \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$ by **L5.2**. Since $\$q_i \in \Gamma_{\Sigma,\theta}$ by construction, we know that $[q_i]_{\Gamma_{\Sigma,\theta}} \in S_{\Gamma_{\Sigma,\theta}}$. Given that $q_i \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ but $q_i \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$, it follows by **L5.10** that $[q_i]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$ but $[q_i]_{\Gamma_{\Sigma,\theta}} \notin |B|_{\Gamma_{\Sigma,\theta}}$, and so $|A|_{\Gamma_{\Sigma,\theta}} \nsubseteq |B|_{\Gamma_{\Sigma,\theta}}$. Thus $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \nvDash A \trianglelefteq B$, and so if $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$, then $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$. We may then conclude that $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$. □

L5.13 $\$A \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$ for all $A \in \texttt{pfs}(\mathcal{L}^+)$.

Proof. Let $A \in \texttt{pfs}(\mathcal{L}^+)$, and assume $\$A \in \Gamma_{\Sigma,\theta}$. By **E1**, $\vdash_{\text{UGSN}} A \trianglelefteq A$, and so $A \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus $[A]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$, and so it follows that $[A]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**. Now assume $x \in |A|_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, there is some $X \in \texttt{pfs}(\mathcal{L}^+)$ such that $x = [X]_{\Gamma_{\Sigma,\theta}}$ where $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. By **SP4**, $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} (A \trianglelefteq X) \vee (X \trianglelefteq \bot)$, and so either $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$ or $X \trianglelefteq \bot \in \Gamma_{\Sigma,\theta}$ by **L5.2**. Since $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \ntrianglelefteq \bot$ by **SP1**, and so $X \trianglelefteq \bot \notin \Gamma_{\Sigma,\theta}$ by **L5.2**. Thus $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$ given the disjunction above. Having already shown that $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \approx A$, and so $X \approx A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus $[X]_{\Gamma_{\Sigma,\theta}} = [A]_{\Gamma_{\Sigma,\theta}}$ by **L5.7**, and so $x = [A]_{\Gamma_{\Sigma,\theta}}$. Since x was arbitrary, $|A|_{\Gamma_{\Sigma,\theta}} = \{[A]_{\Gamma_{\Sigma,\theta}}\}$, and so $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$. We may then conclude that if $\$A \in \Gamma_{\Sigma,\theta}$, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$.

Assuming instead that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$, we know that $|A|_{\Gamma_{\Sigma,\theta}} = \{x\}$ for some $x \in S_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, there is some $X \in \texttt{pfs}(\mathcal{L}^+)$ such that $x = [X]_{\Gamma_{\Sigma,\theta}}$ where both $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. Since $\vdash_{\text{UGSN}} X \trianglelefteq X$ by **E1**, it follows that $X \trianglelefteq X \in \Gamma$ by **L5.2b**, and so $x \in |X|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**. Assume for *reductio* that $A \trianglelefteq X \notin \Gamma_{\Sigma,\theta}$. It follows by **L5.12** that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \nvDash A \trianglelefteq X$, and so $|A|_{\Gamma_{\Sigma,\theta}} \nsubseteq |X|_{\Gamma_{\Sigma,\theta}}$. Thus some $y \in |A|_{\Gamma_{\Sigma,\theta}}$ where $y \notin |X|_{\Gamma_{\Sigma,\theta}}$. Given that $|A|_{\Gamma_{\Sigma,\theta}} = \{x\}$, we know that $y = x$, and so

$x \notin |X|_{\Gamma_{\Sigma,\theta}}$, contradicting the above. By *reductio*, $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGS}} X \approx A$ given that $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. Since $X \approx A \vdash_{\text{UGSN}} \$X \equiv \$A$ by **SP2**, we know $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$X \equiv \$A$. Given that $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$A$, and so $\$A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus if $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$, then $\$A \in \Gamma_{\Sigma,\theta}$, and so $\$A \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$. □

P5.1 If $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, then $\chi \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$ for all $\chi \in \texttt{wfs}(\mathcal{L}^+)$.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ and assume $\Sigma \nvdash_{\text{UGSN}} \theta$. Thus $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent. Assuming $\texttt{comp}^+(\chi) = 0$, either $\chi = A \trianglelefteq B$ or $\chi = \$A$ for some $A, B \in \texttt{pfs}(\mathcal{L}^+)$. If $\chi = A \trianglelefteq B$, then $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$ by **L5.12**, and so $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$. If instead $\chi = \$A$, then $\$A \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$ by **L5.13**, and so $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$. Thus $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$ for any $\chi \in \texttt{wfs}(\mathcal{L}^+)$ where $\texttt{comp}^+(\chi) = 0$. By induction we may show for all $\chi \in \texttt{wfs}(\mathcal{L}^+)$ that $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \chi$. □

6 Infinite Fusion

Given any mereological state space $\langle S, \star \rangle \in \mathbb{M}$ and nonempty finite $X \subseteq S$, there is a unique fusion of the states which belong to X. However, if S is infinite, we may also consider the fusion of infinite subsets of $X \subseteq S$. Accordingly, I will restrict attention to the class of state spaces which is closed under both finite and infinite fusion. More specifically, let an *infinite state space* be any ordered pair $\langle S, \bigsqcup \rangle$ where S is a set closed under the fusion operator $\bigsqcup$ which maps subsets of S to members of S, where $\bigsqcup \varnothing = \square$ is the designated *null state*, and $\bigsqcup S = \blacksquare$ is the designated *full state*. An infinite state space $\langle S, \bigsqcup \rangle$ is *mereological* just in case it also satisfies the following:

Collapse: $\bigsqcup\{s\} = s$ for all $s \in S$.

Associativity: $\bigsqcup\{\bigsqcup E_i : i \in I\} = \bigsqcup \bigcup\{E_i : i \in I\}$ where I indexes each $E_i \subseteq S$.[15]

Let $\mathbb{M}_\infty$ be the class of all infinite mereological state spaces. Now consider:

∞-*Closure:* $\lceil X \rceil = \{\bigsqcup Y : \varnothing \neq Y \subseteq X\}$.

$\mathcal{S}_\infty$-*Propositions:* $\mathbb{P}^\infty_{\mathcal{S}} = \{X \subseteq S : X = \lceil X \rceil\}$.

Given any infinite state space $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup \rangle$, an *infinite unilateral* $\mathcal{S}$*-model* $\mathcal{M}$ *of* $\mathcal{L}^-$ is any ordered triple $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$ where $|p| \in \mathbb{P}^\infty_{\mathcal{S}}$ for all $p \in \mathbb{L}$. Let $\mathcal{C}^\infty_{\mathcal{S}}$ be the class of infinite $\mathcal{S}$-models, and $\mathcal{C}^\infty = \bigcup\{\mathcal{C}^\infty_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$. We may then make the following amendment to the **Unilateral Pre-Semantics**:

($\dot\wedge$) $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

[15] I will let context determine which definition of associativity is intended.

With the exception of the clause given above, the definition of exact verification $\Vdash$ is otherwise unchanged. As before, we may extend the domain of $|\cdot|$:

Infinitary Valuation: $s \in |A|$ *iff* $\mathcal{M}, s \Vdash A$.

Given this definition, the **Unilateral Semantics** may be preserved, thereby defining $\vDash$. As before, $\Gamma \vDash_{\mathcal{C}^\infty} \varphi$ just in case for all $\mathcal{M} \in \mathcal{C}^\infty$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$, where a wfs φ is $\mathcal{C}^\infty$-*valid* just in case $\vDash_{\mathcal{C}^\infty} \varphi$.

In §3, the space of propositions $\mathbb{P}_{\mathcal{S}}$ was shown to be closed under the operators $\wedge$ and $\vee$. We may now show that the space of infinite propositions $\mathbb{P}_{\mathcal{S}}^\infty$ is closed under the infinitary analogues which we may define as follows:

Cartesian Product: Let $\Pi\{X_i : i \in I\}$ be the set of functions $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for all $i \in I$.

Infinite Product: Let $\bigwedge\{X_i : i \in I\} = \{\bigsqcup\{f(i) : i \in I\} : f \in \Pi\{X_i : i \in I\}\}$.

Infinite Sum: Let $\bigvee\{X_i : i \in I\} = \lceil\bigcup\{X_i : i \in I\}\rceil$.

Given any $\mathcal{S} \in \mathbb{M}$ and indexed family of propositions $\{X_i : i \in I\} \in \mathbb{P}_{\mathcal{S}}^\infty$, both:

L6.4 $\bigwedge\{X_i : i \in I\} \in \mathbb{P}_{\mathcal{S}}^\infty$.

L6.5 $\bigvee\{X_i : i \in I\} \in \mathbb{P}_{\mathcal{S}}^\infty$.

Whereas $\mathbb{P}_{\mathcal{S}}$ was only shown to be closed under finitary produce and sum, the results above prove that $\mathbb{P}_{\mathcal{S}}^\infty$ is closed under infinite product and sum.

We may then define finitary analogues of infinite product and sum by letting $X \wedge Y = \bigwedge\{X, Y\}$ and $X \vee Y = \bigvee\{X, Y\}$. Given any $\mathcal{S} \in \mathbb{M}_\infty$, it follows from **L6.6** and **L6.7** that $\mathcal{A}_{\mathcal{S}}^\infty = \langle \mathbb{P}_{\mathcal{S}}^\infty, \wedge, \vee, \mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\!\!\perp \rangle$ is an algebra with the same signature as $\mathcal{A}_{\mathcal{L}^-}$. Moreover, we may show that for any $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}^\infty$, the valuation function $|\cdot| : \mathcal{A}_{\mathcal{L}^-} \to \mathcal{A}_{\mathcal{S}}^\infty$ is a $\mathcal{L}^-$-homomorphism:

L6.9 $|A \wedge B| = |A| \wedge |B|$.

P6.1 $|A| \in \mathbb{P}_{\mathcal{S}}^\infty$.

L6.10 $|A \vee B| = |A| \vee |B|$.

Instead of forming a lattice, $\mathcal{A}_{\mathcal{S}}^\infty$ may be re-described as a *pre-bilattice* consisting of two complete lattices.[16] In particular, we may let $\mathcal{B}_{\mathcal{S}}^\infty = \langle \mathbb{P}_{\mathcal{S}}, \subseteq, \propto \rangle$ where $\subseteq$ is subset inclusion, and $\propto$ is defined by means of the following:

Parthood: $x \sqsubseteq y$ *iff* $\bigsqcup\{x, y\} = y$.

Subsumption: $X \gg Y$ *iff* for all $y \in Y$, there is some $x \in X$ where $x \sqsubseteq y$.

[16] In §7, bilattices are defined in terms of pre-bilattices. See also Ginsberg (1988), Fitting (1991, 2002), and Arieli and Avron (1996). Fine (2017b) also draws this connection.

Subservience: $X \ll Y$ *iff* $x \star y \in Y$ for all $x \in X$ and $y \in Y$.[17]

Containment: $X \propto Y$ *iff* $X \gg Y$ and $X \ll Y$.

We may then prove that for any indexed family of sets $\{X_i : i \in I\} \subseteq \mathbb{P}^{\infty}_{\mathcal{S}}$, both:

L6.6 $\bigwedge\{X_i : i \in I\} = \texttt{lub}^{\propto}\{X_i : i \in I\}$.

L6.7 $\bigvee\{X_i : i \in I\} = \texttt{lub}^{\subseteq}\{X_i : i \in I\}$.

Together with **L6.4** and **L6.5**, it follows from the above that both $\langle \mathbb{P}^{\infty}_{\mathcal{S}}, \propto\rangle$ and $\langle \mathbb{P}^{\infty}_{\mathcal{S}}, \subseteq\rangle$ are complete lattices. Accordingly, $\mathcal{B}^{\infty}_{\mathcal{S}} = \langle \mathbb{P}^{\infty}_{\mathcal{S}}, \propto, \subseteq\rangle$ is a pre-bilattice, paving the way for the introduction of negation in §7.

Given these results, we may extend the *Soundness* and *Completeness* results proven above by restricting consideration to the class of models $\mathcal{C}_{\infty}$ defined over the infinite state spaces in $\mathcal{M}_{\infty}$. In particular:

T3 (*Infinite Fusion*) $\Sigma \vDash_{\mathcal{C}^{\infty}} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$.

Proof. We begin by showing that there are two class functions $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^{\infty}$ and $\mathfrak{B} : \mathcal{C}^{\infty} \to \mathcal{C}$ which preserve logical consequence. More specifically:

P6.3: $\mathcal{M} \vDash \varphi$ iff $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ for all $\mathcal{M} \in \mathcal{C}$ and $\varphi \in \texttt{wfs}(\mathcal{L}^{-})$.

P6.4: $\mathcal{M}_u \vDash \varphi$ iff $\mathcal{M}_u^{\mathfrak{B}} \vDash \varphi$ for all $\mathcal{M}_u \in \mathcal{C}^{\infty}$ and $\varphi \in \texttt{wfs}(\mathcal{L}^{-})$.

Given that $\Sigma \nvDash_{\mathcal{C}} \varphi$, there is some $\mathcal{M} \in \mathcal{C}$ where $\mathcal{M} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M} \nvDash \varphi$. By **P6.3**, $\mathcal{M}^{\mathfrak{F}} \in \mathcal{C}^{\infty}$ where $\mathcal{M}^{\mathfrak{F}} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^{\mathfrak{F}} \nvDash \varphi$, and so $\Sigma \nvDash_{\mathcal{C}^{\infty}} \varphi$. Thus by contraposition, if $\Sigma \vDash_{\mathcal{C}^{\infty}} \varphi$, then $\Sigma \vDash_{\mathcal{C}} \varphi$. Assuming instead $\Sigma \nvDash_{\mathcal{C}^{\infty}} \varphi$, there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M} \nvDash \varphi$. By **P6.4**, $\mathcal{M}^{\mathfrak{B}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{B}} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^{\mathfrak{B}} \nvDash \varphi$, and so $\Sigma \nvDash_{\mathcal{C}} \varphi$. Thus $\Sigma \nvDash_{\mathcal{C}} \varphi$, and so by contraposition, if $\Sigma \vDash_{\mathcal{C}} \varphi$, then $\Sigma \vDash_{\mathcal{C}^{\infty}} \varphi$. Together, $\Sigma \vDash_{\mathcal{C}^{\infty}} \varphi$ iff $\Sigma \vDash_{\mathcal{C}} \varphi$. By **Theorem T1** and **Theorem T2**, we know that $\Sigma \vDash_{\mathcal{C}} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$, and so $\Sigma \vDash_{\mathcal{C}^{\infty}} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$. □

The remainder of the present section will be devoted to proving the results above, many of which will play an important role in the following sections.

L6.1 For any $\mathcal{S} \in \mathbb{M}_{\mathcal{S}}$ and $X \subseteq S$, if $x \in X$ and $\bigsqcup X = y$, then $x \sqsubseteq y$.

Proof. Follows from *Collapse* and *Associativity*. □

L6.2 $x \sqsubseteq x$ for all $\mathcal{S} \in \mathbb{M}_{\mathcal{S}}$ and $x \in S$.

Proof. Immediate from *Collapse*. □

L6.3 For any $\mathcal{S} \in \mathbb{M}_{\mathcal{S}}$ and $x, y, z \in S$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$.

[17] One could define $X \ll Y$ as for all $x \in X$, there is some $y \in Y$ where $x \sqsubseteq y$. These definitions are equivalent provided we require $X, Y \in \mathbb{P}^{\text{C}}$. Compare Fine (2016, p. 207).

Proof. Follows from *Collapse* and *Associativity.* □

L6.4 $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $Y \subseteq \bigwedge\{X_i : i \in I\}$, where $Y = \{y_j : j \in J\}$. By definition, $y_j = \bigsqcup\{f_j(i) : i \in I\}$ where $f_j \in \Pi\{X_i : i \in I\}$ for each $j \in J$. Letting $z_i = \bigsqcup\{f_j(i) : j \in J\}$ for each $i \in I$, it follows that $z_i \in X_i$ since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $\bigsqcup\{z_i : i \in I\} \in \bigwedge\{X_i : i \in I\}$. We may then observe that:

$$\begin{aligned}
\bigsqcup\{z_i : i \in I\} &= \bigsqcup\{\bigsqcup\{f_j(i) : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\{f_j(i) : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\{f_j(i) : i \in I\} : j \in J\} \\
&= \bigsqcup\{\bigsqcup\{f_j(i) : i \in I\} : j \in J\} \\
&= \bigsqcup\{y_j : j \in J\} \\
&= \bigsqcup Y.
\end{aligned}$$

Thus $\bigsqcup Y \in \bigwedge\{X_i : i \in I\}$, and so $\lceil\bigwedge\{X_i : i \in I\}\rceil \subseteq \bigwedge\{X_i : i \in I\}$, where the converse follows by *Collapse*. Thus $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. □

L6.5 $\bigvee\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $Y \subseteq \bigvee\{X_i : i \in I\}$ be nonempty, setting $Y = \{y_j : j \in J\}$. By definition, $y_j = \bigsqcup Z_j$ where $Z_j \subseteq \bigcup\{X_i : i \in I\}$ for each $j \in J$. For each $i \in I$, we may let $x_i = \bigsqcup\bigcup\{Z_j \cap X_i : j \in J\}$, observing that $x_i \in X_i$ since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $\bigsqcup\{x_i : i \in I\} \in \bigvee\{X_i : i \in I\}$. We may then argue as follows:

$$\begin{aligned}
\bigsqcup\{x_i : i \in I\} &= \bigsqcup\{\bigsqcup\bigcup\{Z_j \cap X_i : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\bigcup\{Z_j \cap X_i : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\bigcup\{Z_j \cap X_i : i \in I\} : j \in J\} \\
&= \bigsqcup\bigcup\{Z_j : j \in J\} \\
&= \bigsqcup\{\bigsqcup Z_j : j \in J\} \\
&= \bigsqcup\{y_j : j \in J\} \\
&= \bigsqcup Y.
\end{aligned}$$

Thus $\bigsqcup Y \in \bigvee\{X_i : i \in I\}$, and so $\lceil\bigvee\{X_i : i \in I\}\rceil \subseteq \bigvee\{X_i : i \in I\}$, where the converse holds by *Collapse*. Thus $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. □

L6.6 $\bigwedge\{X_i : i \in I\} = \texttt{lub}^{\propto}\{X_i : i \in I\}$ where $\mathcal{S} \in \mathcal{M}_{\infty}$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}_{\infty}$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^{\infty}_{\mathcal{S}}$, and let $s \in \bigwedge\{X_i : i \in I\}$. It follows that $s = \bigsqcup\{g(i) : i \in I\}$ for some $g : I \to \bigcup\{X_i : i \in I\}$ such that $g(i) \in X_i$ for all $i \in I$. Choose some $i \in I$. If follows that $g(i) \in X_i$, and so $g(i) \sqsubseteq s$. Generalising on s, it follows that $X_i \gg \bigwedge\{X_i : i \in I\}$. Now choose some $x \in X_i$ and $z \in Z$. It follows that $z = \bigsqcup\{f(i) : i \in I\}$

for some $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for all $i \in I$. Since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$, we know that $\bigsqcup\{x, f(i)\} \in X_i$. Consider the definition:

$$f'(j) = \begin{cases} f(j) & \text{if } \; j \neq i \\ \bigsqcup\{x, f(i)\} & \text{otherwise.} \end{cases}$$

It follows that $\bigsqcup\{f'(i) : i \in I\} \in Z$. We may then observe the following:

$$\begin{aligned}
\bigsqcup\{x, z\} &= \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\{f(j) : j \in I\}\} \\
&= \bigsqcup\bigcup\{\{x\}, \{f(j) : j \in I\}\} \\
&= \bigsqcup\bigcup\{\{x, f(i)\}, \{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\{\bigsqcup\{x, f(i)\}, \bigsqcup\{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\bigcup\{\{\bigsqcup\{x, f(i)\}\}, \{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\{f'(j) : j \in I\}.
\end{aligned}$$

Thus $\bigsqcup\{x, z\} \in Z$, and so $X_i \ll \bigwedge\{X_i : i \in I\}$. Together with the above $X_i \propto \bigwedge\{X_i : i \in I\}$. Generalising on $i \in I$, we may conclude that $\bigwedge\{X_i : i \in I\}$ is an upper bound of $\{X_i : i \in I\}$ with respect to $\propto$.

Let $Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$ be an upper bound of $\{X_i : i \in I\}$ with respect to $\propto$. Accordingly, $X_i \propto Z$ for all $i \in I$, and so both $X_i \gg Z$ and $X_i \ll Z$ for all $i \in I$. Choose some $z \in Z$. It follows that for each $i \in I$, there is some $x_i \in X_i$ where $x_i \sqsubseteq z$, and so $\bigsqcup\{x_i, z\} = z$ for each $i \in I$. Accordingly, we may let $f : I \to \bigcup\{X_i : i \in I\}$ be such that $f(i) \in X_i$ for all $i \in I$ where $\bigsqcup\{x_i, z\} = z$. By definition, $\bigsqcup\{f(i) : i \in I\} \in \bigwedge\{X_i : i \in I\}$. Consider:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{f(i) : i \in I\}, z\} &= \bigsqcup\{\bigsqcup\{f(i) : i \in I\}, \bigsqcup\{z\}\} \\
&= \bigsqcup\bigcup\{\{f(i) : i \in I\}, \{z\}\} \\
&= \bigsqcup\bigcup\{\{f(i), z\} : i \in I\} \\
&= \bigsqcup\{\bigsqcup\{f(i), z\} : i \in I\} \\
&= \bigsqcup\{z : i \in I\} \\
&= z.
\end{aligned}$$

By definition, $\bigsqcup\{f(i) : i \in I\} \sqsubseteq z$, and so $\bigwedge\{X_i : i \in I\} \gg Z$ as desired.

Choose instead some $x \in \bigwedge\{X_i : i \in I\}$ and $z \in Z$. It follows that $x = \bigsqcup\{h(i) : i \in I\}$, where $h(i) \in X_i$ for all $i \in I$. Since $X_i \ll Z$ for all $i \in I$, we know that $\bigsqcup\{h(i), z\} \in Z$ for all $i \in I$, and so it follows that $\bigsqcup\{\bigsqcup\{h(i), z\} : i \in I\} \in Z$ given that $Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following:

$$\begin{aligned}
\bigsqcup\{x, z\} &= \bigsqcup\{\bigsqcup\{h(i) : i \in I\}, \bigsqcup\{z\}\} \\
&= \bigsqcup\bigcup\{\{h(i) : i \in I\}, \{z\}\} \\
&= \bigsqcup\bigcup\{\{h(i), z\} : i \in I\} \\
&= \bigsqcup\{\bigsqcup\{h(i), z\} : i \in I\}.
\end{aligned}$$

Thus $\bigsqcup\{x, z\} \in Z$, and so $\bigwedge\{X_i : i \in I\} \ll Z$. Together with the above, $\bigwedge\{X_i : i \in I\} \propto Z$, and so $\bigwedge\{X_i : i \in I\} = \texttt{lub}^{\propto}\{X, Y\}$ as desired. $\square$

L6.7 $\bigvee\{X_i : i \in I\} = \mathtt{lub}^{\subseteq}\{X_i : i \in I\}$ where $\mathcal{S} \in \mathcal{M}_\infty$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}_\infty$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^\infty_{\mathcal{S}}$. Choose some $i \in I$, and $x \in X_i$. It follows that $x \in \bigcup\{X_i : i \in I\}$, and so $\bigsqcup\{x\} \in [\bigcup\{X_i : i \in I\}]$. Thus $x \in \bigvee\{X_i : i \in I\}$ by *Collapse*, and so $X_i \subseteq \bigvee\{X_i : i \in I\}$. Since $i \in I$ was arbitrary, we may conclude that $\bigvee\{X_i : i \in I\}$ is an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$.

Let $Z \in \mathbb{P}^\infty_{\mathcal{S}}$ be an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$. Choose some $x \in \bigvee\{X_i : i \in I\}$. It follows that $x = \bigsqcup Y$ for some nonempty $Y \subseteq \bigcup\{X_i : i \in I\}$. Letting $Y_i = Y \cap X_i$ for each $i \in I$, set $W = \{\bigsqcup Y_i : \varnothing \neq Y_i\}$. Thus for any $w \in W$, we know that $w = \bigsqcup Y_i$ for some $i \in I$, where $Y_i \subseteq X_i$. Since $X_i \in \mathbb{P}^\infty_{\mathcal{S}}$ for all $i \in I$, it follows that $w \in X_i$, and so $w \in Z$ since Z is an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$. Thus $W \subseteq Z$, and so $\bigsqcup W \in Z$ since $Z \in \mathbb{P}^\infty_{\mathcal{S}}$. However:

$$\begin{aligned}
\bigsqcup W &= \bigsqcup\{\bigsqcup Y_i : \varnothing \neq Y_i\}\\
&= \bigsqcup\bigcup\{Y_i : \varnothing \neq Y_i\}\\
&= \bigsqcup\bigcup\{Y \cap X_i : i \in I\}\\
&= \bigsqcup Y\\
&= x.
\end{aligned}$$

Thus $x \in Z$, and so $\bigvee\{X_i : i \in I\} \subseteq Z$ more generally. Given that Z was an arbitrary upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$, we may conclude that $\bigvee\{X_i : i \in I\} = \mathtt{lub}^{\subseteq} X, Y$ as needed. □

L6.8 $U \vee V = U \cup V \cup (U \wedge V)$ for all $U, V \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Similar to *Sum*. □

L6.9 $|A \wedge B| = |A| \wedge |B|$ for all $\mathcal{M} \in \mathcal{C}^\infty$ and $A, B \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. Similar to **L3.4**. □

L6.10 $|A \vee B| = |A| \vee |B|$ for all $\mathcal{M} \in \mathcal{C}^\infty$ and A, B.

Proof. Similar to **L3.5**. □

P6.1 $|A| \in \mathbb{P}^\infty_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}^\infty_{\mathcal{S}}$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^\infty$. By definition, $|p_i| \in \mathbb{P}^\infty_{\mathcal{S}}$ for every $p_i \in \mathbb{L}$, where $|e| \in \mathbb{P}^\infty_{\mathcal{S}}$ for all $e \in \mathbb{E}$. Assume for induction that $|A|, |B| \in \mathbb{P}^\infty_{\mathcal{S}}$. We know that $|A \wedge B| = |A| \wedge |B|$ by **L6.9**, and $|A \vee B| = |A| \vee |B|$ by **L6.10**, and so both $|A| \wedge |B|, |A| \vee |B| \in \mathbb{P}^\infty_{\mathcal{S}}$ by **L6.6** and **L6.7**. It follows by induction that $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$. □

L6.11 $X \wedge \perp\!\!\!\!\perp = X$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$. Consider the following biconditionals:

$$\begin{aligned} s \in X \wedge \perp \ &\textit{iff}\ \ s = \bigsqcup\{x, \square\} \text{ for some } x \in X \\ &\textit{iff}\ \ s = \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\varnothing\} \text{ for some } x \in X \\ &\textit{iff}\ \ s = \bigsqcup\bigcup\{\{x\}, \varnothing\} \text{ for some } x \in X \\ &\textit{iff}\ \ s = \bigsqcup\{x\} \text{ for some } x \in X \\ &\textit{iff}\ \ s \in X. \end{aligned}$$

The above hold by *Collapse* and *Associativity*, and so $X \wedge \perp = X$. □

L6.12 $X \vee \perp = X$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$. By **L6.8**, $X \vee \perp = X \cup \varnothing \cup (X \wedge \varnothing)$. Since $\varnothing \wedge X = \varnothing$, it follows that $X \vee \varnothing = X$. □

L6.13 $X \wedge \varnothing = \varnothing$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Immediate from *Infinite Product.* □

L6.14 $X \vee S = S$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$. By **L6.8**, $X \vee S = X \cup S \cup (X \wedge S)$. Since $X, S \in \mathbb{P}^\infty_\mathcal{S}$, it follows by **L6.4** that $X \wedge S \in \mathbb{P}^\infty_\mathcal{S}$, so $X \wedge S \subseteq S$. Of course, we also know that $X \subseteq S$, and so $S \cup X \cup (S \wedge X) = S$. Thus we may conclude that $S \vee X = X$ a needed. □

L6.15 $(X \wedge Y) \vee (X \wedge Z) \subseteq X \wedge (Y \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}_\mathcal{S}\infty$.

In order to define the function $\mathfrak{F}$, we introduce the following definitions where $\mathcal{S} = \langle S, \star\rangle$ is an arbitrary mereological state space in $\mathbb{M}$:

Ideal: $X \subseteq S$ is an *ideal* in $\langle S, \star\rangle$ *iff* all x, y are such that $x, y \in X$ just in case $x \star y \in X$.

$\mathcal{S}$-Ideals: Let $\mathbb{I}_\mathcal{S}$ be the set of all ideals in $\mathcal{S}$.

X-Ideal: Let $\texttt{ideal}_\mathcal{S}(X) = \bigcap\{Y \in \mathbb{I}_\mathcal{S} : X \subseteq Y\}$.

Parts: Let $\texttt{parts}_\mathcal{S}(x) = \{y \in S : y \star x = x\}$.

We may now prove the following lemmas for an arbitrary state space $\mathcal{S} \in \mathbb{M}$:

L6.16 If $X \subseteq S$, then $\texttt{ideal}_\mathcal{S}(X) \in \mathbb{I}_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}$ where $\mathcal{S} = \langle S, \star\rangle$, and choose some $J \subseteq \mathbb{I}_\mathcal{S}$. We may then let $x, y \in \texttt{ideal}_\mathcal{S}(X)$, choosing some $Y \in \mathbb{I}_\mathcal{S}$ where $X \subseteq Y$. Thus $x, y \in Y$, and so $x \star y \in Y$. Generalising on Y, it follows that $x \star y \in \texttt{ideal}_\mathcal{S}(X)$. Assume instead that $x \star y \in \texttt{ideal}_\mathcal{S}(X)$, choosing an arbitrary $Y \in \mathbb{I}_\mathcal{S}$ where $X \subseteq Y$. It follows that $x \star y \in Y$, and so $x, y \in Y$. Thus $x, y \in \texttt{ideal}_\mathcal{S}(X)$, and so $\texttt{ideal}_\mathcal{S}(X) \in \mathbb{I}_\mathcal{S}$. □

L6.17 For all $X, Y \subseteq S$, if $X \subseteq Y$, then $\mathtt{ideal}_{\mathcal{S}}(X) \subseteq \mathtt{ideal}_{\mathcal{S}}(Y)$.

Proof. Let $X, Y \subseteq S$ where $X \subseteq Y$, and assume $Z \in \mathbb{I}_{\mathcal{S}}$ where $Y \subseteq Z$. It follows that $X \subseteq Z$, and so $\{Z \in \mathbb{I}_{\mathcal{S}} : Y \subseteq Z\} \subseteq \{Z \in \mathbb{I}_{\mathcal{S}} : X \subseteq Z\}$. Thus $\bigcap\{Z \in \mathbb{I}_{\mathcal{S}} : X \subseteq Z\} \subseteq \bigcap\{Z \in \mathbb{I}_{\mathcal{S}} : Y \subseteq Z\}$, or equivalently $\mathtt{ideal}_{\mathcal{S}}(X) \subseteq \mathtt{ideal}_{\mathcal{S}}(Y)$. This concludes the proof. □

L6.18 If $x \in S$, then $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\{x\})$.

Proof. Let $x \in S$, and assume $y \in \mathtt{parts}_{\mathcal{S}}(x)$. It follows that $y \star x = x$. Choose an arbitrary $Y \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq Y$. It follows that $x \in Y$, and so $x \star y \in Y$. Given that Y is an ideal, we know that $x, y \in Y$, and so $y \in Y$. Since $Y \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq Y$ was arbitrary, it follows that $y \in \mathtt{ideal}_{\mathcal{S}}(\{x\})$. Thus $\mathtt{parts}_{\mathcal{S}}(x) \subseteq \mathtt{ideal}_{\mathcal{S}}(\{x\})$ as desired.

In order to show that $\mathtt{parts}_{\mathcal{S}}(x)$ is an ideal, let $y, z \in \mathtt{parts}_{\mathcal{S}}(x)$. It follows that $y \star x = x$ and $z \star x = x$, and so $y \star (z \star x) = x$. By *Associativity*, $(y \star z) \star x = x$, and so $y \star z \in \mathtt{parts}_{\mathcal{S}}(x)$. Assume instead that $y \star z \in \mathtt{parts}_{\mathcal{S}}(x)$. It follows that $(y \star z) \star x = x$, and so both $y \star x = x$ and $z \star x = x$ by *Idempotency*, *Commutativity*, and *Associativity*, and so $y, z \in \mathtt{parts}_{\mathcal{S}}(x)$. Since y and z were arbitrary, $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$.

It follows by *Idempotency* that $x \star x = x$, and so $x \in \mathtt{parts}_{\mathcal{S}}(x)$. Letting $y \in \mathtt{ideal}_{\mathcal{S}}(\{x\})$ be arbitrary, it follows from the above that $y \in \mathtt{parts}_{\mathcal{S}}(x)$ since $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq \mathtt{parts}_{\mathcal{S}}(x)$. Thus $\mathtt{ideal}_{\mathcal{S}}(\{x\}) \subseteq \mathtt{parts}_{\mathcal{S}}(x)$, and so $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\{x\})$. □

L6.19 If $x, y \in S$, then $\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) \subseteq \mathtt{parts}_{\mathcal{S}}(x \star y)$.

Proof. Assume $x, y \in S$, and let $z \in \mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y)$. It follows that either $z \in \mathtt{parts}_{\mathcal{S}}(x)$ or $z \in \mathtt{parts}_{\mathcal{S}}(y)$, and so $z \star x = x$ or $z \star y = y$. In either case $z \star (x \star y) = x \star y$, and so $z \in \mathtt{parts}_{\mathcal{S}}(x \star y)$. Thus we may conclude that $\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) \subseteq \mathtt{parts}_{\mathcal{S}}(x \star y)$. □

L6.20 For all $x, y \in S$, if $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{parts}_{\mathcal{S}}(y)$, then $x = y$.

Proof. Let $x, y \in S$, and assume that $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{parts}_{\mathcal{S}}(y)$. It follows by *Idempotency*, $x \in \mathtt{parts}_{\mathcal{S}}(x)$ and $y \in \mathtt{parts}_{\mathcal{S}}(y)$, and so $x \in \mathtt{parts}_{\mathcal{S}}(y)$ and $y \in \mathtt{parts}_{\mathcal{S}}(x)$. Thus $x \star y = y$ and $y \star x = x$, and so $x = y$. □

L6.21 If $X \subseteq S$, then $X \subseteq \overline{X} \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$.

Proof. Assume $X \subseteq S$, and let $x \in X$. It follows that $x \star x \in \overline{X}$, and so $x \in \overline{X}$ by *Idempotency*. Thus $X \subseteq \overline{X}$.

Assume instead that $x \in \overline{X}$. It follows that $x = y \star z$ for some $y, z \in X$. Choose some $Z \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Z$. Thus $y, z \in Z$, and so $y \star z \in Z$. Generalising on Z, $y \star z \in \mathtt{ideal}_{\mathcal{S}}(X)$, and so $\overline{X} \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$. □

L6.22 If $X \in \mathbb{I}_{\mathcal{S}}$, then $\mathtt{ideal}_{\mathcal{S}}(X) = X$.

Proof. Assume $X \in \mathbb{I}_{\mathcal{S}}$, and let $x \in X$. Choose an arbitrary $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$. It follows that $x \in Y$. Generalising on Y we know that $x \in \mathtt{ideal}_{\mathcal{S}}(X)$, and so it follows that $X \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$.

Assume instead that $x \in \mathtt{ideal}_{\mathcal{S}}(X)$. It follows that $x \in Y$ for all $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$. In particular, $x \in X$ given that $X \in \mathbb{I}_{\mathcal{S}}$. Thus $\mathtt{ideal}_{\mathcal{S}}(X) \subseteq X$, and so $\mathtt{ideal}_{\mathcal{S}}(X) = X$. □

L6.23 If $X \subseteq S$, then $\texttt{ideal}_{\mathcal{S}}(X) = \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Proof. We begin by showing that $\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$. Assume $y, z \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. Thus $y \in \texttt{parts}_{\mathcal{S}}(u)$ and $z \in \texttt{parts}_{\mathcal{S}}(v)$ for some $u, v \in \overline{X}$, and so $y \star u = u$ and $z \star v = v$. Since $\overline{X}$ is closed under fusion, we know that $u \star v \in \overline{X}$, and so $(y \star u) \star (z \star v) \in \overline{X}$. However:

$$\begin{aligned} u \star v &= (y \star u) \star (z \star v) \\ &= (y \star z) \star (u \star v) \end{aligned}$$

Thus $y \star z \in \texttt{parts}_{\mathcal{S}}(u \star v)$ where $u \star v \in \overline{X}$, and so we may conclude that $y \star z \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. In order to establish the converse, assume that $y \star z \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. Thus $y \star z \in \texttt{parts}_{\mathcal{S}}(w)$ where $w \in \overline{X}$, and so $(y \star z) \star w = w$. As in **L6.18**, it follows that $y \star w = w$ and $z \star w = w$, and so both $y, z \in \texttt{parts}_{\mathcal{S}}(w)$ where $w \in \overline{X}$. Thus both $y, z \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$, and so $\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$.

Letting $x \in X$ be arbitrary, $x \star x \in \overline{X}$, and so $x \in \overline{X}$ and $x \in \texttt{parts}_{\mathcal{S}}(x)$. Thus $x \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$, and so $X \subseteq \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. It follows that $\texttt{ideal}_{\mathcal{S}}X \subseteq \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\})$ by **L6.17**, and so $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}) = \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$ given that $\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$. Thus $\texttt{ideal}_{\mathcal{S}}X \subseteq \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Assume instead that $x \in \overline{X}$. By **L6.17**, $\texttt{ideal}_{\mathcal{S}}(\{x\}) \subseteq \texttt{ideal}_{\mathcal{S}}(\overline{X})$. Thus $\texttt{parts}_{\mathcal{S}}(x) \subseteq \texttt{ideal}_{\mathcal{S}}(\overline{X})$ by **L6.18**. Additionally, $\overline{X} \subseteq \texttt{ideal}_{\mathcal{S}}(X)$ by **L6.21**, and so $\texttt{ideal}_{\mathcal{S}}(\overline{X}) \subseteq \texttt{ideal}_{\mathcal{S}}(\texttt{ideal}_{\mathcal{S}}(X))$ by **L6.17**. Thus $\texttt{parts}_{\mathcal{S}}(x) \subseteq \texttt{ideal}_{\mathcal{S}}(X)$ since $\texttt{ideal}_{\mathcal{S}}(\texttt{ideal}_{\mathcal{S}}(X)) = \texttt{ideal}_{\mathcal{S}}(X)$ by **L6.22**. Since $x \in \overline{X}$ was arbitrary, $\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \subseteq \texttt{ideal}_{\mathcal{S}}X$. Together with the above, $\texttt{ideal}_{\mathcal{S}}X = \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. □

L6.24 If $X \subseteq S$, then $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}) = \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Proof. We first show that $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}) = \texttt{ideal}_{\mathcal{S}}(X)$. Choose some $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$, and let $y \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$. It follows that $y \in \texttt{parts}_{\mathcal{S}}(z)$ for some $z \in X$, and so $y \star z = z$ where $z \in Y$. Thus $y \star z \in Y$, and so $y \in Y$. More generally, $\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y$, and so $Y \in \{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\}$. Generalising on Y, we know that $\{Y \in \mathbb{I}_{\mathcal{S}} : X \subseteq Y\} \subseteq \{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\}$, and so $\bigcap\{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\} \subseteq \bigcap\{Y \in \mathbb{I}_{\mathcal{S}} : X \subseteq Y\}$. Equivalently, $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}) \subseteq \texttt{ideal}_{\mathcal{S}}(X)$.

Assume instead that $x \in X$. By *Idempotency*, $x \in \texttt{parts}_{\mathcal{S}}(x)$, and so $x \in \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$. Thus $X \subseteq \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$, and so $\texttt{ideal}_{\mathcal{S}}(X) \subseteq \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\})$ by **L6.17**. Given the above, $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}) = \texttt{ideal}_{\mathcal{S}}(X)$, and so by **L6.23**, $\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}) = \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. □

Given any finite state space $\mathcal{S} \in \mathbb{M}$, we may construct an infinite state space $\mathcal{S}_\infty = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup \rangle$ where $\bigsqcup$ is defined as follows:

Infinite Fusion: $\bigsqcup X = \texttt{ideal}_{\mathcal{S}}(\bigcup X)$ for all $X \subseteq \mathbb{I}_{\mathcal{S}}$.

As I will go on to show, $\mathcal{S}_\infty \in \mathbb{M}_\infty$ for all $\mathcal{S} \in \mathbb{M}$, and so given the definition above, $\mathbb{P}^{\infty}_{\mathcal{S}_\infty} = \{X \subseteq \mathbb{I}_{\mathcal{S}} : X = \lceil X \rceil\}$, which I will write $\mathbb{P}^{\infty}_{\mathcal{S}}$ for simplicity. We may then define a function $\mathfrak{F} : \mathbb{P}_{\mathcal{S}} \to \mathbb{P}^{\infty}_{\mathcal{S}}$ as below:

Forward: $\mathfrak{F}(X) = \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$ for all $X \in \mathbb{P}_{\mathcal{S}}$.

In what follows, I will extend $\mathfrak{F}$ to a function which maps $\mathcal{C}$ into $\mathcal{C}^{\infty}$, where the lemmas given below build towards the construction of an infinite model $\mathcal{M}^{\mathfrak{F}} \in \mathcal{C}^{\infty}$ from any finite model $\mathcal{M} \in \mathcal{C}$, culminating in the proof of **P6.3**.

L6.25 $\bigsqcup U \in \mathbb{I}_{\mathcal{S}}$ for all $U \subseteq \mathbb{I}_{\mathcal{S}}$.

Proof. Let $U \subseteq \mathbb{I}_{\mathcal{S}}$, and assume $x, y \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U)$. Choose some $Z \in \mathbb{I}_{\mathcal{S}}$ where $\bigcup U \subseteq Z$. Thus $x, y \in Z$, and so $x \star y \in Z$. More generally, $x \star y \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U)$. Assume instead that $x \star y \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U)$, letting $V \in \mathbb{I}_{\mathcal{S}}$ where $\bigcup U \subseteq V$. Thus $x \star y \in V$, and so $x, y \in V$. More generally, $x, y \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U)$. Thus $\mathtt{ideal}_{\mathcal{S}}(\bigcup U) \in \mathbb{I}_{\mathcal{S}}$, and so $\bigsqcup U \in \mathbb{I}_{\mathcal{S}}$. □

L6.26 $\bigsqcup\{x\} = x$ for all $x \in \mathbb{I}_{\mathcal{S}}$.

Proof. Let $x \in \mathbb{I}_{\mathcal{S}}$. It follows that $\bigsqcup\{x\} = \mathtt{ideal}_{\mathcal{S}}(\bigcup\{x\}) = \mathtt{ideal}_{\mathcal{S}}(x)$, and so $\bigsqcup\{x\} = x$ by **L6.22**. □

L6.27 $\bigsqcup\{\bigsqcup U_i : i \in I\} = \bigsqcup\bigcup\{U_i : i \in I\}$ if $U_i \subseteq \mathbb{I}_{\mathcal{S}}$ for all $i \in I$.

Proof. Assume $U_i \subseteq \mathbb{I}_{\mathcal{S}}$ for all $i \in I$. We first show that every $Z \in \mathbb{I}_{\mathcal{S}}$ is such that $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$ just in case $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$. Let $Z \in \mathbb{I}_{\mathcal{S}}$ where $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$, and choose some $x \in \bigcup\bigcup\{U_i : i \in I\}$. Thus $x \in \bigcup U_i$ for some $i \in I$. Since $\bigcup U_i \subseteq S$, we know that $\bigcup U_i \subseteq \mathtt{ideal}_{\mathcal{S}}(\bigcup U_i)$ by **L6.21**, and so $x \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U_i)$. Thus $x \in \bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$, and so $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$. By discharge, if $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$, then $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$.

Assume $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$, and let $x \in \bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\}$. Thus $x \in \mathtt{ideal}_{\mathcal{S}}(\bigcup U_i)$ for some $i \in I$, so $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{\bigcup U_i}\}$ by **L6.23**. It follows that $x \in \mathtt{parts}_{\mathcal{S}}(y)$ for some $y \in \overline{\bigcup U_i}$, and so $x \star y = y$ where $y = a \star b$ for some $a, b \in \bigcup U_i$. Thus $a, b \in \bigcup\bigcup\{U_i : i \in I\}$. By assumption, $a, b \in Z$, and so $a \star b \in Z$ since $Z \in \mathbb{I}_{\mathcal{S}}$. Thus $y \in Z$, and so $x \star y \in Z$, given the above. We may then conclude that $x \in Z$, and so $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$. We know by discharge that if $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$, then $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$. Thus $\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$ just in case $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$ for all $Z \in \mathbb{I}_{\mathcal{S}}$ given that $Z \in \mathbb{I}_{\mathcal{S}}$ was arbitrary. Put otherwise:

$$\Big\{Z \in \mathbb{I}_s : \bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z\Big\} = \Big\{Z \in \mathbb{I}_{\mathcal{S}} : \bigcup\bigcup\{U_i : i \in I\} \subseteq Z\Big\}.$$

By then taking the intersection of both sides of the identity above, we may observe that the identities below follow from the definitions:

$$\begin{aligned} \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\}) &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\bigcup\{U_i : i \in I\}) \\ \bigsqcup\{\bigsqcup U_i : i \in I\} &= \bigsqcup\bigcup\{U_i : i \in I\}. \end{aligned}$$

Given this final identity, we may conclude the proof by generalising on the family of sets $U_i \subseteq \mathbb{I}_{\mathcal{S}}$ indexed by I. □

L6.28 For all $X, Y \subseteq \mathbb{I}_{\mathcal{S}}$, if $X \subseteq Y$, then $\lceil X\rceil \subseteq \lceil Y\rceil$.

Proof. Let $X, Y \subseteq \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$, and let $x \in \lceil X \rceil$. Thus some $Z \subseteq X$ where $x = \bigsqcup Z$, and so $Z \subseteq Y$. Thus $x \in \lceil Y \rceil$, and so $\lceil X \rceil \subseteq \lceil Y \rceil$. □

L6.29 $\lceil\lceil U \rceil\rceil = \lceil U \rceil$ for all $U \subseteq \mathbb{I}_{\mathcal{S}}$.

Proof. Let $U \subseteq \mathbb{I}_{\mathcal{S}}$, and assume $u \in \lceil\lceil U \rceil\rceil$. It follows that $u = \bigsqcup V$ for some $V \subseteq \lceil U \rceil$. Let V be indexed by I such that $V = \{v_i : i \in I\}$, and choose some $v_i \in V$. It follows that $v_i = \bigsqcup V_i$ for some $V_i \subseteq U$. Thus:

$$\begin{aligned} u &= \bigsqcup V \\ &= \bigsqcup \{\bigsqcup V_i : i \in I\} \\ &= \bigsqcup \bigcup \{V_i : i \in I\} \end{aligned}$$

The latter identity holds by **L6.27**. Since $V_i \subseteq U$ for all $i \in I$, we know that $\bigcup\{V_i : i \in I\} \subseteq U$, and so $\bigsqcup\bigcup\{V_i : i \in I\} \in \lceil U \rceil$. Thus $u \in \lceil U \rceil$. Generalising on u, it follows that $\lceil\lceil U \rceil\rceil \subseteq \lceil U \rceil$ as desired.

Assume instead that $u \in U$. It follows that $\bigsqcup\{u\} \in \lceil U \rceil$, and so $u \in \lceil U \rceil$ by **L6.26**. Thus $U \subseteq \lceil U \rceil$, and so $\lceil U \rceil \subseteq \lceil\lceil U \rceil\rceil$ by **L6.28**. Together with the above, $\lceil\lceil U \rceil\rceil = \lceil U \rceil$ for all $U \subseteq \mathbb{I}_{\mathcal{S}}$. □

L6.30 $\mathfrak{F}(X) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $X \subseteq S$.

Proof. Let $X \subseteq S$, and $x \in X$. Thus $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\{x\})$ by **L6.18**, and $\texttt{ideal}_{\mathcal{S}}(\{x\}) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**. More generally, $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$ for all $x \in X$, and so $\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \mathbb{I}_{\mathcal{S}}$, where we know by **L6.29** that $\lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil = \lceil\lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil\rceil$. By definition, $\mathfrak{F}(X) = \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$, and so $\mathfrak{F}(X) = \lceil\mathfrak{F}(X)\rceil$. Thus $\mathfrak{F}(X) \in \mathbb{P}^{\infty}_{\mathcal{S}}$, where generalising on $X \subseteq S$ concludes the proof. □

L6.31 $x \in X$ iff $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$ for all $x \in S$ and $X \in \mathbb{P}_{\mathcal{S}}$.

Proof. Let $x \in S$ and $X \in \mathbb{P}_{\mathcal{S}}$ where $x \in X$. Of course, it follows that $\texttt{parts}_{\mathcal{S}}(x) \in \{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$, where we know by definition that $\bigsqcup\{\texttt{parts}_{\mathcal{S}}(x)\} \in \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$. Since $x \in S$, it follows that $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\{x\})$ by **L6.18**, and $\texttt{ideal}_{\mathcal{S}}(\{x\}) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**, and so $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$. Thus $\bigsqcup\{\texttt{parts}_{\mathcal{S}}(x)\} = \texttt{parts}_{\mathcal{S}}(x)$ follows by **L6.26**, and so $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$.

Assume instead that $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$. Thus there is some $Y \subseteq \{\texttt{parts}_{\mathcal{S}}(y) : y \in X\}$ where $\texttt{parts}_{\mathcal{S}}(x) = \bigsqcup Y$. By definition, $\bigsqcup Y = \texttt{ideal}_{\mathcal{S}}(\bigcup Y)$, and so $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\bigcup Y)$. We may then let $Z = \{z \in X : \texttt{parts}_{\mathcal{S}}(z) \in Y\}$, and so $\bigcup Y = \bigcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in Z\}$. Accordingly, $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in Z\})$, and so by **L6.24** we know that $\texttt{parts}_{\mathcal{S}}(x) = \bigcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\}$. Of course, $x \in \texttt{parts}_{\mathcal{S}}(x)$ by *Idempotency*, and so $x \in \bigcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\}$. Thus $x = \texttt{parts}_{\mathcal{S}}(y)$ for some $y \in \overline{Z}$, and so $x \star y = y$. Letting $u \in \texttt{parts}_{\mathcal{S}}(x)$, it follows that $u \star x = x$, and so we may reason as follows:

$$\begin{aligned} u \star y &= u \star (x \star y) \\ &= (u \star x) \star y \\ &= x \star y \\ &= y \end{aligned}$$

Thus it follows that $u \in \texttt{parts}_{\mathcal{S}}(y)$. Generalising on u, it follows that $\texttt{parts}_{\mathcal{S}}(x) \subseteq \texttt{parts}_{\mathcal{S}}(y)$. However, given that $y \in \overline{Z}$, we also know that $\texttt{parts}_{\mathcal{S}}(y) \subseteq \bigcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\} = \texttt{parts}_{\mathcal{S}}(x)$, from which it follows that $\texttt{parts}_{\mathcal{S}}(y) \subseteq \texttt{parts}_{\mathcal{S}}(x)$. Given the above, $\texttt{parts}_{\mathcal{S}}(x) = \texttt{parts}_{\mathcal{S}}(y)$, and so $x = y$ by **L6.20**. Thus we may conclude that $x \in \overline{Z}$.

Recall from above that $Z = \{z \in X : \texttt{parts}_{\mathcal{S}}(z) \in Y\}$. It follows that $Z \subseteq X$, and so $\overline{Z} \subseteq \overline{X}$. However, we know by assumption that $X \in \mathbb{P}_{\mathcal{S}}$, and so $\overline{X} = X$. Thus $\overline{Z} \subseteq X$, and so $x \in X$ given the above. Thus $x \in X$ just in case $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$. ☐

L6.32 $X \subseteq Y$ iff $\mathfrak{F}(X) \subseteq \mathfrak{F}(Y)$, $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $X \subseteq Y$ for some $X, Y \in \mathbb{P}_{\mathcal{S}}$. We may then observe that $\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\}$, and so it follows by **L6.28** that $\lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil \subseteq \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil$. Thus $\mathfrak{F}(X) \subseteq \mathfrak{F}(Y)$.

Assume $X \nsubseteq Y$ instead, and so $x \notin Y$ for some $x \in X$. Thus we may conclude by **L6.31** that $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$ but $\texttt{parts}_{\mathcal{S}}(x) \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil$. It follows that $\mathfrak{F}(X) \nsubseteq \mathfrak{F}(Y)$. ☐

L6.33 $\mathfrak{F}(\{x\}) = \{\texttt{parts}_{\mathcal{S}}(x)\}$ for all $x \in S$.

Proof. Let $x \in S$, so $\mathfrak{F}(\{x\}) = \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in \{x\}\}\rceil = \lceil\{\texttt{parts}_{\mathcal{S}}(x)\}\rceil$, where $\lceil\{\texttt{parts}_{\mathcal{S}}(x)\}\rceil = \{\bigsqcup\{\texttt{parts}_{\mathcal{S}}(x)\}\} = \{\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x)\})\}$. Of course, $\{\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x)\})\} = \{\texttt{ideal}_{\mathcal{S}}(\texttt{parts}_{\mathcal{S}}(x))\}$. By **L6.18** and **L6.16**, we also know that $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$, and so it follows by **L6.22** that $\texttt{ideal}_{\mathcal{S}}(\texttt{parts}_{\mathcal{S}}(x)) = \texttt{parts}_{\mathcal{S}}(x)$. Thus $\mathfrak{F}(\{x\}) = \{\texttt{parts}_{\mathcal{S}}(x)\}$. ☐

L6.34 $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) = \mathfrak{F}(X \wedge Y)$ for all $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Let $X, Y \in \mathbb{P}_{\mathcal{S}}$, and assume $Z \in \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$. Thus $Z = \bigsqcup\{U, V\}$ for some $U \in \mathfrak{F}(X)$ and $V \in \mathfrak{F}(Y)$, and so $Z = \texttt{ideal}_{\mathcal{S}}(U \cup V)$. We also know that $U = \bigsqcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X'\}$ and $V = \bigsqcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in Y'\}$ for some $X' \subseteq X$ and $Y' \subseteq Y$. If follows by definition that both:

$$U = \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X'\}),$$
$$V = \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in Y'\}).$$

Given these identities, we may observe that the identities below follow by definition or assumption with exception of $(*)$ which is given by **L6.24**:

$$\begin{aligned}
Z &= \texttt{ideal}_{\mathcal{S}}(U \cup V)\\
&= \texttt{ideal}_{\mathcal{S}}[\texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X'\}) \cup \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in Y'\})]\\
(*) \quad &= \texttt{ideal}_{\mathcal{S}}[\bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{X'}\} \cup \bigcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in \overline{Y'}\}]\\
&= \texttt{ideal}_{\mathcal{S}}[\bigcup\{\texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y) : x \in \overline{X'}, y \in \overline{Y'}\}]\\
&= \bigsqcup\{\texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y) : x \in \overline{X'}, y \in \overline{Y'}\}
\end{aligned}$$

Let $z \in \texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y)$ for some $x \in \overline{X'}$ and $y \in \overline{Y'}$. By **L6.19**, $z \in \texttt{parts}_{\mathcal{S}}(x \star y)$, where $x \in X$ and $y \in Y$ given that $\overline{X'} \subseteq \overline{X} = X$

and $\overline{Y'} \subseteq \overline{Y} = Y$. Thus $z \in \mathtt{parts}_{\mathcal{S}}(x \star y)$ for $x \in X$ and $y \in Y$, and so $z \in \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}$. Generalising on z, it follows that:

$$\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in X, y \in Y\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\},$$
$$\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in X, y \in Y\} \in \lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil.$$

Given the above, $Z \in \lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil$. However, by definition, $\lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil = \mathfrak{F}(X \wedge Y)$, and so $Z \in \mathfrak{F}(X \wedge Y)$. Thus it follows that $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) \subseteq \mathfrak{F}(X \wedge Y)$.

Assume instead that $Z \in \mathfrak{F}(X \wedge Y)$. It follows that $Z = \bigsqcup J$ for some $J \subseteq \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}$. Thus there is some $K \subseteq X \wedge Y$ where $J = \{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\}$. Hence it follows that:

$$\begin{aligned} Z &= \bigsqcup J \\ &= \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\} \\ &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\}) \\ (\dagger) \quad &= \bigcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in \overline{K}\} \\ (\ddagger) \quad &= \mathtt{ideal}_{\mathcal{S}}(K) \end{aligned}$$

Whereas (†) is given by **L6.24**, (‡) holds by **L6.23**, where the remaining identities follow from the above by definition. Consider the definitions:

$$\begin{aligned} X' &= \{x \in X : \exists y \in Y \text{ where } x \star y \in K\}, \\ Y' &= \{y \in Y : \exists x \in X \text{ where } x \star y \in K\}, \\ U &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}), \\ V &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\}). \end{aligned}$$

We may then show that $\mathtt{ideal}_{\mathcal{S}}(K) = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ where $U \subseteq X$ and $V \subseteq Y$. Let $k \in K$. Since $K \subseteq X \wedge Y$, we know that $k = x \star y$ where $x \in X$ and $y \in Y$, and so $x \in X'$ and $y \in Y'$. Given that $x \in \mathtt{parts}_{\mathcal{S}}(x)$ and $y \in \mathtt{parts}_{\mathcal{S}}(y)$ by *Idempotency*, both $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}$ and $y \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\}$. By **L6.21**, $x \in U$ and $y \in V$. Letting $Z \in \mathbb{I}_{\mathcal{S}}$ where $U \cup V \subseteq Z$, it follows that $x, y \in Z$, and so $x \star y \in Z$. Generalising on Z, we may conclude that $x \star y \in \mathtt{ideal}_{\mathcal{S}}(U \cup V)$, and so $k \in \mathtt{ideal}_{\mathcal{S}}(U \cup V)$. Thus $K \subseteq \mathtt{ideal}_{\mathcal{S}}(U \cup V)$, and so by **L6.17** $\mathtt{ideal}_{\mathcal{S}}(K) \subseteq \mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(U \cup V))$. Since $\mathtt{ideal}_{\mathcal{S}}(U \cup V) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**, it follows that $\mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(U \cup V)) = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ by **L6.22**, and so $\mathtt{ideal}_{\mathcal{S}}(K) \subseteq \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ given the above.

In order to establish the converse inclusion, choose some $W \in \mathbb{I}_{\mathcal{S}}$ where $K \subseteq W$, and let $w \in U \cup V$. Consider the following cases:

Case 1: Assume $w \in U$. By **L6.24**, $U = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X'}\}$, and so $w \in \mathtt{parts}_{\mathcal{S}}(x)$ for some $x \in \overline{X'}$. It follows that $w \star x = x$ where $x = a \star b$ for some $a, b \in X'$. Thus there are some u and v such that $a \star u \in K$ and $b \star v \in K$, and so both $a \star u, b \star v \in W$ given that $K \subseteq W$. Since $W \in \mathbb{I}_{\mathcal{S}}$, both $a, b \in W$, and so $a \star b \in W$. Given the above, we may conclude that $w \star x \in W$, and so $w \in W$ as desired. *Case 2:* Assume $w \in V$. By **L6.24**, $V = \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{Y'}\}$, and so $w \in \mathtt{parts}_{\mathcal{S}}(y)$ for some $y \in \overline{Y'}$. It follows that $w \star y = y$ where $y = c \star d$ for some

$c, d \in Y'$. Thus there are some u and v such that $c \star u \in K$ and $d \star v \in K$, and so both $c \star u, d \star v \in W$ given that $K \subseteq W$. Since $W \in \mathbb{I}_{\mathcal{S}}$, both $c, d \in W$, and so $c \star d \in W$. Given the above, we may conclude that $w \star y \in W$, and so $w \in W$ as desired.

Since $w \in W$ in both of the cases above, it follows that $U \cup V \subseteq W$, and so $\{W \in \mathbb{I}_{\mathcal{S}} : K \subseteq W\} \subseteq \{W \in \mathbb{I}_{\mathcal{S}} : U \cup V \subseteq W\}$. Thus we know that $\bigcap\{W \in \mathbb{I}_{\mathcal{S}} : U \cup V \subseteq W\} \subseteq \bigcap\{W \in \mathbb{I}_{\mathcal{S}} : K \subseteq W\}$, and so $\texttt{ideal}_{\mathcal{S}}(U \cup V) \subseteq \texttt{ideal}_{\mathcal{S}}(K)$ by definition. Together with the inclusion given above, it follow that $\texttt{ideal}_{\mathcal{S}}(K) = \texttt{ideal}_{\mathcal{S}}(U \cup V)$.

Thus we have $Z = \texttt{ideal}_{\mathcal{S}}(K) = \texttt{ideal}_{\mathcal{S}}(U \cup V) = \bigsqcup\{U, V\}$. Since $X' \subseteq X$ and $Y' \subseteq Y$, both $\{\texttt{parts}_{\mathcal{S}}(x) : x \in X'\} \subseteq \{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$ and $\{\texttt{parts}_{\mathcal{S}}(y) : y \in Y'\} \subseteq \{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\}$. We may then observe:

$$U = \bigsqcup\{\texttt{parts}_{\mathcal{S}}(x) : x \in X'\} \in \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil = \mathfrak{F}(X),$$
$$V = \bigsqcup\{\texttt{parts}_{\mathcal{S}}(y) : y \in Y'\} \in \lceil\{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil = \mathfrak{F}(Y).$$

Having shown that $Z = \bigsqcup\{U, V\}$ where $U \in \mathfrak{F}(X)$ and $V \in \mathfrak{F}(Y)$, it follows that $Z \in \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$. Since $Z \in \mathfrak{F}(X \wedge Y)$ was arbitrary, it follows that $\mathfrak{F}(X \wedge Y) \subseteq \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$, and so $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) = \mathfrak{F}(X \wedge Y)$. □

L6.35 $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \mathfrak{F}(X \vee Y)$ for all $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Letting $X, Y \in \mathbb{P}_{\mathcal{S}}$, the inclusions given on the left hold by *Sum*, where the inclusions on the right follow by **L6.32**:

$$\begin{aligned} X &\subseteq X \vee Y & \mathfrak{F}(X) &\subseteq \mathfrak{F}(X \vee Y) \\ Y &\subseteq X \vee Y & \mathfrak{F}(Y) &\subseteq \mathfrak{F}(X \vee Y) \\ X \wedge Y &\subseteq X \vee Y & \mathfrak{F}(X \wedge Y) &\subseteq \mathfrak{F}(X \vee Y). \end{aligned}$$

Since $\mathfrak{F}(X \wedge Y) = \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$ by **L6.34**, we may conclude from the above that $\mathfrak{F}(X) \cup \mathfrak{F}(Y) \cup (\mathfrak{F}(X) \wedge \mathfrak{F}(Y)) \subseteq \mathfrak{F}(X \vee Y)$. Thus by **L6.8**, we may conclude that $\mathfrak{F}(X) \vee \mathfrak{F}(Y) \subseteq \mathfrak{F}(X \vee Y)$.

Choose some $x \in X$. Since $\{\texttt{parts}_{\mathcal{S}}(x)\} \subseteq \{\texttt{parts}_{\mathcal{S}}(x) : x \in X\}$, we know by definiton that $\bigsqcup\{\texttt{parts}_{\mathcal{S}}(x)\} \in \mathfrak{F}(X)$. We also know by **L6.16** that $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$, and so $\bigsqcup\{\texttt{parts}_{\mathcal{S}}(x)\} = \texttt{parts}_{\mathcal{S}}(x)$ by **L6.26**. Thus $\texttt{parts}_{\mathcal{S}}(x) \in \mathfrak{F}(X)$, and so $\{\texttt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \mathfrak{F}(X)$ follows by generalising on $x \in X$. By analogous arguments:

$$\begin{aligned} \{\texttt{parts}_{\mathcal{S}}(y) : y \in Y\} &\subseteq \mathfrak{F}(Y) \\ \{\texttt{parts}_{\mathcal{S}}(z) : z \in X \wedge Y\} &\subseteq \mathfrak{F}(X \wedge Y). \end{aligned}$$

However, $\mathfrak{F}(X \wedge Y) = \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$ by **L6.34**, and so it follows that:

$$\{\texttt{parts}_{\mathcal{S}}(z) : z \in X \cup Y \cup (X \wedge Y)\} \subseteq \mathfrak{F}(X) \cup \mathfrak{F}(Y) \cup (\mathfrak{F}(X) \wedge \mathfrak{F}(Y)).$$

Thus $\{\texttt{parts}_{\mathcal{S}}(z) : z \in X \vee Y\} \subseteq \mathfrak{F}(X) \vee \mathfrak{F}(Y)$ by *Sum* and **L6.8**. By **L6.28**, $\lceil\{\texttt{parts}_{\mathcal{S}}(z) : z \in X \vee Y\}\rceil \subseteq \lceil\mathfrak{F}(X) \vee \mathfrak{F}(Y)\rceil$, and so by definition $\mathfrak{F}(X \vee Y) \subseteq \lceil\mathfrak{F}(X) \vee \mathfrak{F}(Y)\rceil$. However, $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil$, and so $\mathfrak{F}(X \vee Y) \subseteq \lceil\lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil\rceil$. Since $X, Y \in \mathbb{P}_{\mathcal{S}}$, we know by **L6.30** that both $\mathfrak{F}(X), \mathfrak{F}(Y) \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and so $\mathfrak{F}(X), \mathfrak{F}(Y) \subseteq \mathbb{I}_{\mathcal{S}}$. Thus $\mathfrak{F}(X) \cup \mathfrak{F}(Y) \subseteq \mathbb{I}_{\mathcal{S}}$, and so $\lceil\lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil\rceil = \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil$ by **L6.29**. We may then conclude that $\mathfrak{F}(X \vee Y) \subseteq \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil = \mathfrak{F}(X) \vee \mathfrak{F}(Y)$, and so by the inclusion established above, we know that $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \mathfrak{F}(X \vee Y)$. □

Given these results, we are now in a position to extend the mapping $\mathfrak{F}$ from propositions to models. In particular, consider the following:

$\mathfrak{F}$-Map: Let $\mathcal{M}^{\mathfrak{F}} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup, |\cdot|^{\mathfrak{F}} \rangle$ where $\mathcal{M} = \langle S, \star, |\cdot| \rangle \in \mathcal{C}$ and $|\cdot|^{\mathfrak{F}} = \mathfrak{F}(|\cdot|)$.

I will then show that $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^{\infty}$ is a strong homomorphism with respect to logical consequence. To begin with, consider the following lemmas.

L6.36 $\mathcal{M}^{\mathfrak{F}} \in \mathcal{C}^{\infty}$ for all $\mathcal{M} \in \mathcal{C}$.

Proof. Let $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ where $\mathcal{S} = \langle S, \star \rangle$. By **L6.25**, $\mathcal{I} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup \rangle$ is an infinite state space, where $\mathcal{I} \in \mathbb{M}_{\infty}$ by **L6.26** and **L6.27**. Letting $p \in \mathbb{L}$ be arbitrary, we may then observe that $|p| \subseteq S$, and so $\mathfrak{F}(|p|) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ by **L6.30**. Generalising on $p \in \mathbb{L}$, it follows that $\mathcal{M}^{\mathfrak{F}} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup, |\cdot|^{\mathfrak{F}} \rangle \in \mathcal{C}^{\infty}$. □

L6.37 $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \mathtt{pfs}(\mathcal{L}^{-})$.

Proof. The proof goes by induction on complexity where the base case is given by *$\mathfrak{F}$-Map*. Assume for induction that $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \mathtt{pfs}(\mathcal{L}^{-})$ where $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}(A)$ with $\mathtt{comp}(A) = n$. It follows that either $A = B \wedge C$ or $A = B \vee C$, where in either case $|B|^{\mathfrak{F}} = \mathfrak{F}(|B|)$ and $|C|^{\mathfrak{F}} = \mathfrak{F}(|C|)$ by hypothesis.

Case 1: Assume $A = B \wedge C$. We may then reason as follows:

$$
\begin{array}{rl}
s \in |B \wedge C|^{\mathfrak{F}} & \textit{iff} \;\; \mathcal{M}^{\mathfrak{F}}, s \Vdash B \wedge C \\
& \textit{iff} \;\; s = \bigsqcup\{a, b\} \text{ where } \mathcal{M}^{\mathfrak{F}}, a \Vdash B \text{ and } \mathcal{M}^{\mathfrak{F}}, b \Vdash C \\
& \textit{iff} \;\; s = \bigsqcup\{a, b\} \text{ where } a \in |B|^{\mathfrak{F}} \text{ and } b \in |C|^{\mathfrak{F}} \\
(1) & \textit{iff} \;\; s = \bigsqcup\{a, b\} \text{ where } a \in \mathfrak{F}(|B|) \text{ and } b \in \mathfrak{F}(|C|) \\
& \textit{iff} \;\; s \in \mathfrak{F}(|B|) \wedge \mathfrak{F}(|C|). \\
(2) & \textit{iff} \;\; s \in \mathfrak{F}(|B| \wedge |C|). \\
(3) & \textit{iff} \;\; s \in \mathfrak{F}(|B \wedge C|).
\end{array}
$$

Whereas (1) follows by hypothesis, (2) is given by **L6.34**, and (3) follows from **L3.4**. All of the other biconditionals are immediate from the definitions. Thus $|B \wedge C|^{\mathfrak{F}} = \mathfrak{F}(|B \wedge C|)$ and so $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$.

Case 2: Assume $B = C \vee C$ instead. Thus if follows that:

$$
\begin{array}{rl}
s \in |B \vee C|^{\mathfrak{F}} & \textit{iff} \;\; \mathcal{M}^{\mathfrak{F}}, s \Vdash B \vee C \\
& \textit{iff} \;\; \mathcal{M}^{\mathfrak{F}}, s \Vdash B, \text{ or } \mathcal{M}^{\mathfrak{F}}, s \Vdash C, \text{ or } \mathcal{M}^{\mathfrak{F}}, s \Vdash B \wedge C \\
& \textit{iff} \;\; s \in |B|^{\mathfrak{F}}, \text{ or } s \in |C|^{\mathfrak{F}}, \text{ or } s \in |B \wedge C|^{\mathfrak{F}} \\
(1) & \textit{iff} \;\; s \in \mathfrak{F}(|B|), \text{ or } s \in \mathfrak{F}(|C|), \text{ or } s \in \mathfrak{F}(|B \wedge C|) \\
(2) & \textit{iff} \;\; s \in \mathfrak{F}(|B|) \cup \mathfrak{F}(|C|) \cup (\mathfrak{F}(|B|) \wedge \mathfrak{F}(|C|)) \\
(3) & \textit{iff} \;\; s \in \mathfrak{F}(|B|) \vee \mathfrak{F}(|C|). \\
(4) & \textit{iff} \;\; s \in \mathfrak{F}(|B| \vee |C|). \\
(5) & \textit{iff} \;\; s \in \mathfrak{F}(|B \vee C|).
\end{array}
$$

As above, (1) is given by hypothesis, whereas (2) follows by the argument given in *Case 1*, (3) holds by **L6.8**, (4) is given by **L6.35**, and (5) follows from **L3.3**. Since the other biconditionals all hold by definition, we may conclude that $|B \vee C|^{\mathfrak{F}} = \mathfrak{F}(|B \vee C|)$ and so $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$.

Since $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ holds in both of the cases above, we may conclude by induction that $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \texttt{pfs}(\mathcal{L}^-)$ as desired. □

P6.2 $|A|^{\mathfrak{F}} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{pfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M} \in \mathcal{C}$. By **P3.1**, $|A| \in \mathbb{P}_{\mathcal{S}}$, and so $|A| \subseteq S$. Thus $\mathfrak{F}(|A|) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ by **L6.30**, and so $|A|^{\mathfrak{F}} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ follows by **L6.37**. □

P6.3 $\mathcal{M} \vDash \varphi$ iff $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$, for all $\mathcal{M} \in \mathcal{C}$ and $\varphi \in \texttt{wfs}(\mathcal{L}^-)$.

Proof. Let $\mathcal{M} = \langle S, \star, |\cdot|\rangle$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$. The proof goes by induction on the complexity. Assume $\varphi \in \texttt{wfs}(\mathcal{L}^-)$ where $\texttt{comp}^+(\varphi) = 0$. Thus either $\varphi = \$A$ or $\varphi = A \trianglelefteq B$ for some $A, B \in \texttt{pfs}(\mathcal{L}^-)$.

Case (\$)*:* Assume $\varphi = \$A$ for $A \in \texttt{pfs}(\mathcal{L}^-)$. Thus $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so $s \in |A|$ just in case $\texttt{parts}_{\mathcal{S}}(s) \in \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in |A|\}\rceil$ for any $s \in S$ by **L6.31**. Since $\mathfrak{F}(|A|) = \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in |A|\}\rceil$ by definition, and $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ by **L6.37**, we may conclude the following:

$$s \in |A| \ \ \textit{iff} \ \ \texttt{parts}_{\mathcal{S}}(s) \in |A|^{\mathfrak{F}}. \tag{$*$}$$

Assume $\mathcal{M} \vDash \varphi$. It follows that $\mathcal{M} \vDash \$A$, and so $|A| = \{a\}$ for some $a \in S$. By **L6.33**, $\mathfrak{F}(\{a\}) = \{\texttt{parts}_{\mathcal{S}}(a)\}$, and so $|A|^{\mathfrak{F}} = \{\texttt{parts}_{\mathcal{S}}(a)\}$. Thus $\mathcal{M}^{\mathfrak{F}} \vDash \$A$, and so we may conclude that $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$.

Assume $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ instead. Thus $\mathcal{M}^{\mathfrak{F}} \vDash \$A$, and so $|A|^{\mathfrak{F}} = \{b\}$ for some $b \in \mathbb{I}_{\mathcal{S}}$. Since $|A|^{\mathfrak{F}} = \lceil\{\texttt{parts}_{\mathcal{S}}(x) : x \in |A|\}\rceil$ by definition, we may conclude that $\{b\} = \{\bigsqcup Y : \varnothing \neq Y \subseteq \{\texttt{parts}_{\mathcal{S}}(x) : x \in |A|\}\}$. It follows that $|A| \neq \varnothing$, for otherwise $\{b\} = \varnothing$, and so there is some $x \in |A|$. In order to prove uniqueness, assume $y \in |A|$. By $(*)$, both $\texttt{parts}_{\mathcal{S}}(x), \texttt{parts}_{\mathcal{S}}(y) \in |A|^{\mathfrak{F}}$, and so $\texttt{parts}_{\mathcal{S}}(x) = \texttt{parts}_{\mathcal{S}}(y)$ given that $|A|^{\mathfrak{F}} = \{b\}$. Thus $x = y$ by **L6.20**, and so every $y \in |A|$ is such that $x = y$. Thus $|A| = \{x\}$, and so $\mathcal{M} \vDash \$A$. It follows that $\mathcal{M} \vDash \varphi$, and so together with the above, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$.

Case ($\trianglelefteq$)*:* Assume $\varphi = A \trianglelefteq B$ for $A, B \in \texttt{pfs}(\mathcal{L}^-)$. By **P3.1** both $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$, and so $|A| \subseteq |B|$ just in case $\mathfrak{F}(|A|) \subseteq \mathfrak{F}(|B|)$ by **L6.32**. By **L6.37**, $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ and $|B|^{\mathfrak{F}} = \mathfrak{F}(|B|)$, and so we know:

$$|A| \subseteq |B| \ \ \textit{iff} \ \ |A|^{\mathfrak{F}} \subseteq |B|^{\mathfrak{F}}. \tag{$*$}$$

Thus $\mathcal{M} \vDash A \trianglelefteq B$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash A \trianglelefteq B$, or equivalently, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ as desired. Given the above, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$ such that $\texttt{comp}^+(\varphi) = 0$.

Since $\mathcal{M}$ agree $\mathcal{M}^{\mathfrak{F}}$ on all atomic sentences of $\mathcal{L}^-$, it follows by routine induction that $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$. □

By **P6.3**, $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^{\infty}$ is a strong homomorphism with respect to $\vDash$. It remains to construct a function $\mathfrak{B} : \mathcal{C}^{\infty} \to \mathcal{C}$ with the same structure preserving property. Given any $\mathcal{M}_u \in \mathcal{C}^{\infty}$ where $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u\rangle$ let:

Backwards: $\mathfrak{B}(X) = X$ for all $X \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

$\mathfrak{B}$-*Map:* Let $\mathcal{M}_u^{\mathfrak{B}} = \langle S, \star, |\cdot|_u^{\mathfrak{B}}\rangle$ where $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u\rangle \in \mathcal{C}^\infty$ and $|\cdot|_u^{\mathfrak{B}} = \mathfrak{B}(|\cdot|_u)$.

The following lemma proves that $\mathfrak{B}$ preserves semantic entailment in evaluating wfs of $\mathcal{L}^-$ at finite models in $\mathcal{C}$ rather than infinite models in $\mathcal{C}^\infty$.

L6.38 $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ for all $\mathcal{M}_u \in \mathcal{C}^\infty$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. The proof goes by induction on complexity where the base case is given by $\mathfrak{B}$-*Map*. Assume for induction that $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$ where $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}(A)$ with $\mathtt{comp}(A) = n$. It follows that either $A = B \wedge C$ or $A = B \vee C$, where in either case $|B|_u^{\mathfrak{B}} = \mathfrak{B}(|B|_u)$ and $|C|_u^{\mathfrak{B}} = \mathfrak{B}(|C|_u)$ by hypothesis.

Case 1: Assume $A = B \wedge C$. We may then reason as follows:

$$
\begin{aligned}
s \in |B \wedge C|_u^{\mathfrak{B}} \quad & \textit{iff} \;\; \mathcal{M}_u^{\mathfrak{B}}, s \Vdash B \wedge C \\
& \textit{iff} \;\; s = a \star b \text{ where } \mathcal{M}_u^{\mathfrak{B}}, a \Vdash B \text{ and } \mathcal{M}_u^{\mathfrak{B}}, b \Vdash C \\
& \textit{iff} \;\; s = a \star b \text{ where } a \in |B|_u^{\mathfrak{B}} \text{ and } b \in |C|_u^{\mathfrak{B}} \\
(*) \quad & \textit{iff} \;\; s = a \star b \text{ where } a \in \mathfrak{B}(|B|_u) \text{ and } b \in \mathfrak{B}(|C|_u) \\
& \textit{iff} \;\; s = \bigsqcup\{a, b\} \text{ where } a \in |B|_u \text{ and } b \in |C|_u \\
& \textit{iff} \;\; \mathcal{M}_u, s \Vdash B \wedge C \\
& \textit{iff} \;\; s \in |B \wedge C|_u.
\end{aligned}
$$

Whereas $(*)$ follows by hypothesis, all of the other biconditionals are immediate from the definitions. Thus $|B \wedge C|_u^{\mathfrak{B}} = |B \wedge C|_u = \mathfrak{B}(|B \wedge C|_u)$ and so $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ as desired.

Case 2: Assume $B = C \vee C$ instead. Thus if follows that:

$$
\begin{aligned}
s \in |B \vee C|_u^{\mathfrak{B}} \quad & \textit{iff} \;\; \mathcal{M}_u^{\mathfrak{B}}, s \Vdash B \vee C \\
& \textit{iff} \;\; \mathcal{M}_u^{\mathfrak{B}}, s \Vdash B, \text{ or } \mathcal{M}_u^{\mathfrak{B}}, s \Vdash C, \text{ or } \mathcal{M}_u^{\mathfrak{B}}, s \Vdash B \wedge C \\
& \textit{iff} \;\; s \in |B|_u^{\mathfrak{B}}, \text{ or } s \in |C|_u^{\mathfrak{B}}, \text{ or } s \in |B \wedge C|_u^{\mathfrak{B}} \\
(*) \quad & \textit{iff} \;\; s \in \mathfrak{B}(|B|_u), \text{ or } s \in \mathfrak{B}(|C|_u), \text{ or } s \in \mathfrak{B}(|B \wedge C|_u) \\
& \textit{iff} \;\; \mathcal{M}_u, s \Vdash B, \text{ or } \mathcal{M}_u, s \Vdash C, \text{ or } \mathcal{M}_u, s \Vdash B \wedge C \\
& \textit{iff} \;\; \mathcal{M}_u, s \Vdash B \vee C \\
& \textit{iff} \;\; s \in |B \vee C|_u.
\end{aligned}
$$

As above, (*) is given by hypothesis, whereas the other biconditionals follow by definition. Thus $|B \vee C|_u^{\mathfrak{B}} = |B \vee C|_u = \mathfrak{B}(|B \vee C|_u)$ and so $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$. Since $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ holds in both of the cases above, it follows by induction that $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$. □

P6.4 $\mathcal{M}_u \vDash \varphi$ iff $\mathcal{M}_u^{\mathfrak{B}} \vDash \varphi$, for all $\mathcal{M}_u \in \mathcal{C}^\infty$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M}_u \in \mathcal{C}^\infty$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. The proof goes by induction on the complexity of $\varphi \in (\mathcal{L}^-)$. Assume to start that $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$ where $\mathtt{comp}^+(\varphi) = 0$. Thus either $\varphi = \$A$ or $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. I will consider these cases in order.

Case (\$)*:* Assume $\varphi = \$A$ for $A \in \mathtt{pfs}(\mathcal{L}^-)$. Since $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ by the **L6.38**, where $|A|_u = \mathfrak{B}(|A|_u)$ by $\mathfrak{B}$-*Map*, it follows that $|A|_u = \{s\}$

for some $s \in S$ just in case $|A|_u^{\mathfrak{B}} = \{s\}$ for some $s \in S$. Thus $\mathcal{M}_u \vDash \$A$ just in case $\mathcal{M}_u^{\mathfrak{B}} \vDash \$A$, and so $\mathcal{M}_u \vDash \varphi$ just in case $\mathcal{M}_u^{\mathfrak{B}} \vDash \varphi$.

Case ($\trianglelefteq$)*:* Assume $\varphi = A \trianglelefteq B$ for $A, B \in \texttt{pfs}(\mathcal{L}^-)$. Thus by **L6.38**, both $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ and $|B|_u^{\mathfrak{B}} = \mathfrak{B}(|B|_u)$, where $|A|_u = \mathfrak{B}(|A|_u)$ and $|B|_u = \mathfrak{B}(|B|_u)$ by $\mathfrak{B}$-*Map*, we know $|A|_u \subseteq |B|_u$ just in case $|A|_u^{\mathfrak{B}} \subseteq |B|_u^{\mathfrak{B}}$. It follows that $\mathcal{M}_u \vDash A \trianglelefteq B$ just in case $\mathcal{M}_u^{\mathfrak{B}} \vDash A \trianglelefteq B$, and so we may conclude that $\mathcal{M}_u \vDash \varphi$ just in case $\mathcal{M}_u^{\mathfrak{B}} \vDash \varphi$ as desired.

Given that $\mathcal{M}$ agree $\mathcal{M}^{\mathfrak{B}}$ on all atomic sentences of $\mathcal{L}^-$, it follows by a standard induction that $\mathcal{M} \vDash \varphi$ iff $\mathcal{M}^{\mathfrak{B}} \vDash \varphi$ for all $\varphi \in \texttt{wfs}(\mathcal{L}^-)$. ☐

7 Negation

Recall that negation was excluded from the pfs of $\mathcal{L}^-$. I will now extend the results proven above to a logic in which this simplification is dropped. Accordingly, I will take '$\mathcal{V}$' and '$\top\!\!\!\bot$' to abbreviate '$\neg\mathcal{T}$' and '$\neg\bot$' respectively, letting $\mathcal{L}$ be the result of excluding $\mathcal{V}$ and $\top\!\!\!\bot$ from the primitive symbols of $\mathcal{L}^-$ while including '$\neg$'. We may then amend the formation rules as follows:

($\neg$) If A is a pfs of $\mathcal{L}$, then $\neg A$ is a pfs of $\mathcal{L}$.

Let $\texttt{pfs}^\neg(\mathcal{L})$ be the set of all pfss of $\mathcal{L}$ once the clause for negation has been added to the formation rules given above, replacing '$\mathcal{L}^-$' with '$\mathcal{L}$' throughout. We may then let $\texttt{wfs}^\neg(\mathcal{L})$ be the set of all wfss of $\mathcal{L}$ generated recursively from $\texttt{pfs}^\neg(\mathcal{L})$ *via* $\texttt{atoms}^\neg(\mathcal{L})$ in the same manner as before. In addition to the axioms and rules of inference for UGSN, we may now include the following:

Negation Axioms

NA1 $A \trianglelefteq \neg\neg A$.

NA2 $\neg\neg A \trianglelefteq A$.

NA3 $\neg A \wedge \neg B \trianglelefteq \neg(A \vee B)$.

NA4 $\neg(A \vee B) \trianglelefteq \neg A \wedge \neg B$.

NA5 $\neg A \vee \neg B \trianglelefteq \neg(A \wedge B)$.

NA6 $\neg(A \wedge B) \trianglelefteq \neg A \vee \neg B$.

Let $\vdash_{\text{UGS}}$ be the smallest relation closed under the axioms and rules given both here and in §2. A *theorem* of *The Specific Logic of Unilateral Ground* (UGS) is any $\varphi \in \texttt{wfs}^\neg(\mathcal{L})$ where $\vdash_{\text{UGS}} \varphi$. We may then derive the following:

Negation Equivalences

E15 $\neg A \wedge \neg B \approx \neg(A \vee B)$.

E17 $A \approx \neg\neg A$.

E16 $\neg A \vee \neg B \approx \neg(A \wedge B)$.

It remains to adapt the semantics to accommodate negation. In particular, every proposition will be shown to have a unique inverse.

Following Fine (2017b,c), I will take propositions to be ordered pairs of sets of states, where the states belonging to the first set may be referred to as the *exact verifiers* and the second set as the *exact falsifiers*. More specifically, for any $\mathcal{S} \in \mathbb{M}_\infty$, we may define the space of *bilateral propositions* as follows:

Bilateral $\mathcal{S}$-Propositions: $\mathbb{P}^{\pm}_{\mathcal{S}} = \{\langle V, F\rangle : V, F \in \mathbb{P}^{\infty}_{\mathcal{S}}\}$.

Given any $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup\rangle$, a *bilateral $\mathcal{S}$-model* is any ordered triple $\mathcal{M} = \langle S, \bigsqcup, |\cdot|\rangle$ where every $p_i \in \mathbb{L}$ is such that $|p_i| = \langle |p_i|^+, |p_i|^-\rangle$ for some $|p_i|^\pm \in \mathbb{P}_{\mathcal{S}}$. Letting $\mathcal{C}^{\pm}_{\mathcal{S}}$ be the class of all bilateral $\mathcal{S}$-models for any $\mathcal{S} \in \mathbb{M}_\infty$, and $\mathcal{C}^{\pm} = \bigcup\{\mathcal{C}^{\pm}_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$, I will adopt the following bilateral pre-semantics:

Bilateral Pre-Semantics:

$(\mathcal{T})^+$	$\mathcal{M}, s \Vdash \mathcal{T}$ *iff* $s = s$.	$(\bot)^+$	$\mathcal{M}, s \Vdash \bot$ *iff* $s \neq s$.
$(\mathcal{T})^-$	$\mathcal{M}, s \dashv\mid \mathcal{T}$ *iff* $s = \blacksquare$.	$(\bot)^-$	$\mathcal{M}, s \dashv\mid \bot$ *iff* $s = \square$.
$(p_i)^+$	$\mathcal{M}, s \Vdash p_i$ *iff* $s \in \lvert p_i\rvert^+$.	$(\neg)^+$	$\mathcal{M}, s \Vdash \neg A$ *iff* $\mathcal{M}, s \dashv\mid A$.
$(p_i)^-$	$\mathcal{M}, s \dashv\mid p_i$ *iff* $s \in \lvert p_i\rvert^-$.	$(\neg)^-$	$\mathcal{M}, s \dashv\mid \neg A$ *iff* $\mathcal{M}, s \Vdash A$.

$(\wedge)^+$ $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

$(\wedge)^-$ $\mathcal{M}, s \dashv\mid A \wedge B$ *iff* $\mathcal{M}, s \dashv\mid A$ or $\mathcal{M}, s \dashv\mid B$ or $\mathcal{M}, s \dashv\mid A \vee B$.

$(\vee)^+$ $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

$(\vee)^-$ $\mathcal{M}, s \dashv\mid A \vee B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \dashv\mid A$ and $\mathcal{M}, t \dashv\mid B$.

Whereas $\mathcal{T}$ is verified by every state and falsified by the full state which we may think of as most impossible state, $\bot$ is verified by no state and falsified by the null state which may think of as obtaining trivially. We may then observe that the negation clauses formalise the idea that the exact verifiers for $\neg A$ are exact falsifiers for A, and the exact falsifiers for $\neg A$ are exact verifiers for A. For instance, the state of affairs of my sitting exactly verifies 'I am sitting' just as much as it exactly falsifies its negation 'I am not sitting'. Whereas $(\wedge)^+$ and $(\vee)^+$ are the same as the clauses $(\wedge)$ and $(\vee)$ given in the **Unilateral Pre-Semantics** above, we find in $(\vee)^-$ that only the fusion of falsifiers for both disjuncts will falsify the disjunction as a whole, and in $(\wedge)^-$ an exact falsifier for either conjunct will exactly falsify the conjunction as a whole, as well as a fusion of exact falsifiers for each of the conjuncts.

Having stated the definition of $\Vdash$ and $\dashv\mid$, we may then expand the domain of the functions $|\cdot|^\pm$ to include all pfs A of $\mathcal{L}$ as follows:

Bilateral Valuation: $s \in |A|^+$ *iff* $\mathcal{M}, s \Vdash A$.

$s \in |A|^-$ *iff* $\mathcal{M}, s \dashv\mid A$.

We may then adopt $|A| = \langle |A|^+, |A|^-\rangle$ as standard notation for the proposition that the pfs A of $\mathcal{L}$ expresses in $\mathcal{M}$. The semantic clauses for the wfss of $\mathcal{L}$ are identical to those given in §3 with the exception of replacing $|\cdot|$ in **Unilateral Semantics** with $|\cdot|^+$. The atomic clauses will then read:

Bilateral Semantics:

$(\unlhd)^{\pm}$ $\mathcal{M} \vDash A \unlhd B$ *iff* $|A|^{+} \subseteq |B|^{+}$.

$(\$)^{\pm}$ $\mathcal{M}^{+} \vDash \$A$ *iff* there is exactly one $s \in |A|^{+}$.

The semantic clauses for $\neg$, $\wedge$, and $\vee$ are the same as given before. Accordingly, we may let $\Gamma \vDash_{\mathcal{C}^{\pm}} \varphi$ just in case for all $\mathcal{M} \in \mathcal{C}^{\pm}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$. As usual, a wfs φ is $\mathcal{C}^{\pm}$-*valid* just in case $\vDash_{\mathcal{C}^{\pm}} \varphi$.

Given the addition of falsifiers to the semantics, we are now in a position to introduce the following operations where $P_i = \langle P_i^{+}, P_i^{-} \rangle$ for all $i \in I$:

Bilateral Product: Let $\bigwedge\{P_i : i \in I\} = \langle \bigwedge\{P_i^{+} : i \in I\}, \bigvee\{P_i^{-} : i \in I\}\rangle$.

Bilateral Sum: Let $\bigvee\{P_i : i \in I\} = \langle \bigvee\{P_i^{+} : i \in I\}, \bigwedge\{P_i^{-} : i \in I\}\rangle$.

As before, it will be useful to consider binary analogues of bilateral product and sum, letting $P \wedge Q = \bigwedge\{P, Q\}$ and $P \vee Q = \bigvee\{P, Q\}$. We may then introduce the following unary inversion operator, where $X, Y \in \mathbb{P}_{\mathcal{S}}^{\infty}$ are arbitrary:

Inversion: $\neg\langle X, Y\rangle = \langle Y, X\rangle$.

The reason for introducing falsifiers in addition to verifiers is best exhibited by the definition given above, since a set of verifiers on their own cannot determine a unique set of verifiers for its inverse. By taking propositions to be ordered pairs of a set of verifiers and set of falsifiers, inverses are uniquely determined by permuting the sets of verifiers and falsifiers.

Given the expanded formation rules for the pfss of $\mathcal{L}$, we may observe that $\mathcal{A}_{\mathcal{L}} = \langle \mathtt{pfs}^{\neg}(\mathcal{L}), \neg, \wedge, \vee, \mathcal{T}, \bot\rangle$ and $\mathcal{A}_{\mathcal{S}}^{\pm} = \langle \mathbb{P}_{\mathcal{S}}^{\pm}, \neg, \wedge, \vee, \mathcal{T}, \bot\rangle$ are both algebras with the same signature for any $\mathcal{S} \in \mathbb{M}_{\infty}$, where every model $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}^{\pm}$ induces an $\mathcal{L}$-homomorphism $|\cdot| : \mathcal{A}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{S}}^{\pm}$ since $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$:

P7.1 $|A| \in \mathbb{P}_{\mathcal{S}}^{\pm}$.

L7.1 $|\neg A| = \neg|A|$.

L7.2 $|A \wedge B| = |A| \wedge |B|$.

L7.3 $|A \vee B| = |A| \vee |B|$.

The results above show that the structure encoded by the sentential operators $\wedge$, $\vee$, and $\neg$ is preserved by every model $\mathcal{M} \in \mathcal{C}^{\pm}$. Additionally, it is easy to show that for any $\mathcal{M} \in \mathcal{C}^{\pm}$, the bilateral extremal propositions are as follows:

Bilateral Extremal Propositions

L7.4: $|\mathcal{T}| = \langle S, \{\blacksquare\}\rangle$.

L7.5: $|\bot| = \langle \varnothing, \{\square\}\rangle$.

L7.6: $|\mathcal{V}| = \langle \{\blacksquare\}, S\rangle$.

L7.7: $|\perp\!\!\!\!\underline{\ \ }| = \langle \{\square\}, \varnothing\rangle$.

I will now turn to establish the following theorem, extending the *Soundness* and *Completeness* results proven above to the expanded system UGS.

T4 (*Negation Extension*) $\Sigma \vDash_{\mathcal{C}^{\pm}} \varphi$ iff $\Sigma \vdash_{\text{UGS}} \varphi$.

Proof. In order to extend both *Soundness* and *Completeness* to UGS, I will introduce the functions $\texttt{neg} : \texttt{wfs}^{\neg}(\mathcal{L}) \to \texttt{wfs}(\mathcal{L}^{-})$ and $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$ where $\texttt{neg}(\Sigma) = \{\texttt{neg}(\sigma) : \sigma \in \Sigma\}$, proving the following results:

P7.2: $\Sigma \vdash_{\text{UGS}} \varphi$ iff $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$ for all $\Sigma \cup \{\varphi\} \subseteq \texttt{wfs}^{\neg}(\mathcal{L})$.

P7.3: $\mathcal{M} \vDash \texttt{neg}(\varphi)$ iff $\mathcal{M}^{\mathfrak{N}} \vDash \varphi$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$ and $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$.

P7.4: $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$ is a surjection.

Given the above, we may then establish both the *Soundness* and *Completeness* for UGS over $\mathcal{C}^{\pm}$ by means of the following argument:

$\Sigma \nvdash_{\text{UGS}} \varphi$ *iff*$_1$ $\texttt{neg}(\Sigma) \nvdash_{\text{UGSN}} \texttt{neg}(\varphi)$
iff$_2$ $\texttt{neg}(\Sigma) \nvDash_{\mathcal{C}^{\infty}} \texttt{neg}(\varphi)$
iff$_3$ some $\mathcal{M} \in \mathcal{C}^{\infty}$ is such that $\mathcal{M} \vDash \texttt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but $\mathcal{M} \nvDash \texttt{neg}(\varphi)$
iff$_4$ some $\mathcal{M}_u \in \mathcal{C}^{\pm}$ is such that $\mathcal{M}_u \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}_u \nvDash \varphi$
iff$_5$ $\Sigma \nvDash_{\mathcal{C}^{\pm}} \varphi$.

Here (1) is given by **P7.2**, (2) follows from **Theorem T3**, both (3) and (5) hold by definition. It remains to establish (4).

Assume for discharge that there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ such that $\mathcal{M} \vDash \texttt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but where $\mathcal{M} \nvDash \texttt{neg}(\varphi)$. It follows by **P7.3** that $\mathcal{M}^{\mathfrak{N}} \vDash \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^{\mathfrak{N}} \nvDash \varphi$, where existentially generalising on $\mathcal{M}^{\mathfrak{N}} \in \mathcal{C}^{\pm}$ completes the forward direction. Assume instead that there is some $\mathcal{M}_u \in \mathcal{C}^{\pm}$ such that $\mathcal{M}_u \vDash \sigma$ for all $\sigma \in \Sigma$ but where $\mathcal{M}_u \nvDash \varphi$. By **P7.4**, we know that there there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M}^{\mathfrak{N}} = \mathcal{M}_u$. Thus it follows by **P7.3** that $\mathcal{M} \vDash \texttt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but where $\mathcal{M} \nvDash \texttt{neg}(\varphi)$, thereby completing the reverse direction. We may then conclude that $\Sigma \vDash_{\mathcal{C}^{\pm}} \varphi$ iff $\Sigma \vdash_{\text{UGS}} \varphi$. □

The remainder of the present section will be devoted to proving the results stated above. The following section will draw connections with bilattice theory.

R2$^{\pm}$ If $\Gamma \vdash_{\text{UGS}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGS}} \varphi_{[A/p]}$. (*Uniform Substitution*)

Proof. The proof is similar to **AR2**, where identical reasoning applies to the axioms **NA2** – **NA6** and the axioms and rules in UGSN. □

E15 $\vdash_{\text{UGSN}} \neg A \wedge \neg B \approx \neg(A \vee B)$.

Proof. Follows from **NA3** and **NA4**. □

E16 $\vdash_{\text{UGSN}} \neg A \vee \neg B \approx \neg(A \wedge B)$.

Proof. Follows from **NA5** and **NA6**. □

E17 $\vdash_{\text{UGSN}} A \approx \neg\neg A$.

Proof. Follows from **NA1** and **NA2**. □

L7.1 $|\neg A| = \langle |A|^-, |A|^+\rangle$ if $\mathcal{M} \in \mathcal{C}^\pm$ and $|A| \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Given the semantics for negation, we know that:

$$\begin{aligned} s \in |\neg A|^+ \;\; & \textit{iff}\;\; \mathcal{M}, s \Vdash \neg A \\ & \textit{iff}\;\; \mathcal{M}, s \dashv\Vert A \\ & \textit{iff}\;\; s \in |A|^-. && [^+_\neg] \end{aligned}$$

$$\begin{aligned} s \in |\neg A|^- \;\; & \textit{iff}\;\; \mathcal{M}, s \dashv\Vert \neg A \\ & \textit{iff}\;\; \mathcal{M}, s \Vdash A \\ & \textit{iff}\;\; s \in |A|^+. && [^-_\neg] \end{aligned}$$

It follows that $|\neg A|^+ = |A|^-$ and $|\neg A|^- = |A|^+$, and so we may conclude that $|\neg A| = \langle |A|^-, |A|^+\rangle$ as desired. □

L7.2 $|A \wedge B| = \langle |A|^+ \wedge |B|^+, |A|^- \vee |B|^-\rangle$ if $\mathcal{M} \in \mathcal{C}^\pm$ and $|A|, |B| \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^\pm$ and $|A|, |B| \in \mathbb{P}^\pm_\mathcal{S}$. We first demonstrate that $|A \wedge B|^+ = |A|^+ \wedge |B|^+$ and $|A \vee B|^- = |A|^- \wedge |B|^-$ as follows:

$$\begin{aligned} s \in |A \wedge B|^+ \;\; & \textit{iff}\;\; \mathcal{M}, s \Vdash A \wedge B \\ & \textit{iff}\;\; s = \bigsqcup\{d, t\} \text{ where } \mathcal{M}, d \Vdash A \text{ and } \mathcal{M}, t \Vdash B \\ & \textit{iff}\;\; s = \bigsqcup\{d, t\} \text{ where } d \in |A|^+ \text{ and } t \in |B|^+ \\ & \textit{iff}\;\; s \in |A|^+ \wedge |B|^+. && [^+_\wedge] \\ s \in |A \vee B|^- \;\; & \textit{iff}\;\; \mathcal{M}, s \dashv\Vert A \vee B \\ & \textit{iff}\;\; s = \bigsqcup\{d, t\} \text{ where } \mathcal{M}, d \dashv\Vert A \text{ and } \mathcal{M}, t \dashv\Vert B \\ & \textit{iff}\;\; s = \bigsqcup\{d, t\} \text{ where } d \in |A|^- \text{ and } t \in |B|^- \\ & \textit{iff}\;\; s \in |A|^- \wedge |B|^-. && [^-_\vee] \end{aligned}$$

The biconditionals in the arguments above hold by definition. Given that $|A|, |B| \in \mathbb{P}^\pm_\mathcal{S}$, it follows that $|A|^-, |B|^- \in \mathbb{P}^\infty_\mathcal{S}$. Consider the following:

$$\begin{aligned} s_i \in |A \wedge B|^- \;\; & \textit{iff}\;\; \mathcal{M}, s_i \dashv\Vert A \wedge B \\ & \textit{iff}\;\; \mathcal{M}, s_i \dashv\Vert A, \text{ or } \mathcal{M}, s_i \dashv\Vert B, \text{ or } \mathcal{M}, s_i \dashv\Vert A \vee B \\ & \textit{iff}\;\; s_i \in |A|^-, \text{ or } s_i \in |B|^-, \text{ or } s_i \in |A \vee B|^- \\ & \textit{iff}\;\; s_i \in |A|^- \cup |B|^- \cup |A \vee B|^- \\ (\dagger) \;\; & \textit{iff}\;\; s_i \in |A|^- \cup |B|^- \cup (|A|^- \wedge |B|^-) \\ (\ddagger) \;\; & \textit{iff}\;\; s \in |A|^- \vee |B|^-. && [^-_\wedge] \end{aligned}$$

Each of the biconditionals above hold by definition with the exception of (†) which follows from $[^-_\vee]$, and (‡) which is given by **L6.8**. Thus it follows that $|A \wedge B|^- = |A|^- \vee |B|^-$. Since $|A \wedge B| = \langle |A \wedge B|^+, |A \wedge B|^-\rangle$ by **Valuation**, we know that $|A \wedge B| = \langle |A|^+ \wedge |B|^+, |A|^- \vee |B|^-\rangle$. □

L7.3 $|A \vee B| = \langle |A|^+ \vee |B|^+, |A|^- \wedge |B|^-\rangle$ if $\mathcal{M} \in \mathcal{C}^\pm$ and $|A|, |B| \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Assume $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. We know $|A \vee B| = \langle |A \vee B|^+, |A \vee B|^- \rangle$ by **Valuation**, and $|A|^+, |B|^+ \in \mathbb{P}^{\infty}_{\mathcal{S}}$ given that $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. Thus:

$$\begin{aligned}
s \in |A \vee B|^+ \quad & \textit{iff} \quad \mathcal{M}, s \Vdash A \vee B \\
& \textit{iff} \quad \mathcal{M}, s \Vdash A, \text{ or } \mathcal{M}, s \Vdash B, \text{ or } \mathcal{M}, s \Vdash A \wedge B \\
& \textit{iff} \quad s \in |A|^+, \text{ or } s \in |B|^+, \text{ or } s \in |A \wedge B|^+ \\
& \textit{iff} \quad s \in |A|^+ \cup |B|^+ \cup |A \wedge B|^+ \\
(\dagger) \quad & \textit{iff} \quad s \in |A|^+ \cup |B|^+ \cup (|A|^+ \wedge |B|^+) \\
(\ddagger) \quad & \textit{iff} \quad s \in |A|^+ \vee |B|^+. \qquad [^+_\vee]
\end{aligned}$$

The biconditionals above hold by definition with the exception of (†) which follows from $[^+_\wedge]$, and (‡) which is given by **L6.8**. Thus it follows that $|A \vee B|^+ = |A|^+ \vee |B|^+$, and since $|A \vee B|^- = |A|^- \wedge |B|^-$ as shown by $[^-_\vee]$ in **L7.2**, we know that $|A \vee B| = \langle |A|^+ \vee |B|^+, |A|^- \wedge |B|^- \rangle$. □

P7.1 $|A| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ and $A \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^{\pm}$. By definition, $|p| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for every $p \in \mathbb{L}$, where $|e| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for all $e \in \mathbb{E}$ by **Bilateral Extremal Propositions**, thereby establishing the base case. Assume $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for induction. Thus $|A|^{\pm}, |B|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$, where we know by **L7.1**, **L7.2**, and **L7.3** that:

$$\begin{aligned}
|\neg A| &= \langle |A|^-, |A|^+ \rangle \\
|A \wedge B| &= \langle |A|^+ \wedge |B|^+, |A|^- \vee |B|^- \rangle \\
|A \vee B| &= \langle |A|^+ \vee |B|^+, |A|^- \wedge |B|^- \rangle.
\end{aligned}$$

We know by **L6.4** and **L6.5** that $|A|^+ \wedge |B|^+$, $|A|^- \vee |B|^-$, $|A|^+ \vee |B|^+$, and $|A|^- \wedge |B|^-$ are members of $\mathbb{P}^{\infty}_{\mathcal{S}}$, and so $|\neg A|, |A \wedge B|, |A \vee B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows by induction that $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \texttt{pfs}^{\neg}(\mathcal{L})$. □

L7.4 $|\mathcal{T}| = \langle S, \{\blacksquare\} \rangle$.

Proof. Follows from the **Bilateral Pre-Semantics**. □

L7.5 $|\bot| = \langle \varnothing, \{\square\} \rangle$.

Proof. Follows from the **Bilateral Pre-Semantics**. □

L7.6 $|\mathcal{V}| = \langle \{\blacksquare\}, S \rangle$.

Proof. By abbreviation, $|\mathcal{V}| = |\neg\mathcal{T}|$, where $|\neg\mathcal{T}| = \neg|\mathcal{T}|$ by **L7.1**. Thus $\neg|\mathcal{T}| = \neg\langle S, \{\blacksquare\} \rangle$ by **L7.4**, where $\neg\langle S, \{\blacksquare\} \rangle = \langle \{\blacksquare\}, S \rangle$ holds by definition. We may then conclude that $|\mathcal{V}| = \langle \{\blacksquare\}, S \rangle$ as needed. □

L7.7 $|\bot\!\!\!\!-| = \langle \{\square\}, \varnothing \rangle$.

Proof. By abbreviation, $|\bot\!\!\!\!-| = |\neg\bot|$, where $|\neg\bot| = \neg|\bot|$ by **L7.1**. Thus $\neg|\bot| = \neg\langle \varnothing, \{\square\} \rangle$ by **L7.5**, where $\neg\langle \varnothing, \{\square\} \rangle = \langle \{\square\}, \varnothing \rangle$ by definition. We may then conclude that $|\bot\!\!\!\!-| = \langle \{\square\}, \varnothing \rangle$ as needed. □

We may now introduce the function $\texttt{neg} : \texttt{wfs}^{\neg}(\mathcal{L}) \to \texttt{wfs}(\mathcal{L}^{-})$ which works by first distributing negation over disjunction and conjunction in the sub-pfs of any $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$, after which the literals which occur therein are mapped to unique sentence letters in $\mathbb{L}$. In particular, consider the following:

$$\begin{array}{rcl}
\texttt{neg}(\mathcal{T}) & = & \mathcal{T} \\
\texttt{neg}(\bot) & = & \bot \\
\texttt{neg}(\neg\mathcal{T}) & = & \mathcal{V} \\
\texttt{neg}(\neg\bot) & = & \underline{\bot} \\
\texttt{neg}(p_i) & = & p_{2i} \\
\texttt{neg}(\neg p_i) & = & p_{2i+1} \\
\texttt{neg}(\neg\neg A) & = & \texttt{neg}(A) \\
\texttt{neg}(A \wedge B) & = & \texttt{neg}(A) \wedge \texttt{neg}(B) \\
\texttt{neg}(A \vee B) & = & \texttt{neg}(A) \vee \texttt{neg}(B) \\
\texttt{neg}(\neg(A \vee B)) & = & \texttt{neg}(\neg A) \wedge \texttt{neg}(\neg B) \\
\texttt{neg}(\neg(A \wedge B)) & = & \texttt{neg}(\neg A) \vee \texttt{neg}(\neg B) \\
\texttt{neg}(A \trianglelefteq B) & = & \texttt{neg}(A) \trianglelefteq \texttt{neg}(B) \\
\texttt{neg}(A \ntrianglelefteq B) & = & \texttt{neg}(A) \ntrianglelefteq \texttt{neg}(B) \\
\texttt{neg}(\$A) & = & \$\,\texttt{neg}(A) \\
\texttt{neg}(\neg\$A) & = & \neg\$\,\texttt{neg}(A) \\
\texttt{neg}(\neg\neg\varphi) & = & \neg\neg\,\texttt{neg}(\varphi) \\
\texttt{neg}(\varphi \wedge \psi) & = & \texttt{neg}(\varphi) \wedge \texttt{neg}(\psi) \\
\texttt{neg}(\varphi \vee \psi) & = & \texttt{neg}(\varphi) \vee \texttt{neg}(\psi) \\
\texttt{neg}(\neg(\varphi \vee \psi)) & = & \neg(\texttt{neg}(\varphi) \vee \texttt{neg}(\psi)) \\
\texttt{neg}(\neg(\varphi \wedge \psi)) & = & \neg(\texttt{neg}(\varphi) \wedge \texttt{neg}(\psi)).
\end{array}$$

The function neg works by replacing the elements of $\texttt{pfs}^{\neg}(\mathcal{L})$ which occur in any $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$ with $\texttt{pfs}(\mathcal{L}^{-})$, thereby returning a sentence which belongs to $\texttt{wfs}(\mathcal{L}^{-})$. We may then show that $\texttt{neg} : \texttt{wfs}^{\neg}(\mathcal{L}) \to \texttt{wfs}(\mathcal{L}^{-})$ as follows.

L7.8 $\texttt{neg}(A) \in \texttt{pfs}(\mathcal{L}^{-})$ for all $A \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Follows by a routine induction proof. □

L7.9 $\texttt{neg}(\varphi) \in \texttt{wfs}(\mathcal{L}^{-})$ for all $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$.

Proof. Follows from **L7.8** by a routine induction proof. □

Having defined the function $\texttt{neg} : \texttt{wfs}^{\neg}(\mathcal{L}) \to \texttt{wfs}(\mathcal{L}^{-})$, we may now draw on uniform substitution in order to define a function $\texttt{gen} : \texttt{wfs}(\mathcal{L}^{-}) \to \texttt{wfs}^{\neg}(\mathcal{L})$:

Substitution:
$$\texttt{gen}(\mathcal{A}) = \mathcal{A}_{[\neg\bot/\underline{\bot}][\neg\mathcal{T}/\mathcal{V}][p_i/p_{2i}][\neg p_i/p_{2i+1}]}$$
$$\texttt{gen}(\varphi) = \varphi_{[\neg\bot/\underline{\bot}][\neg\mathcal{T}/\mathcal{V}][p_i/p_{2i}][\neg p_i/p_{2i+1}]}.$$

The following two lemmas show that $\texttt{gen}(\texttt{neg}(\varphi))$ and φ are ground-theoretically equivalent for all $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$.

L7.10 $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ for all $A \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Let $A \in \texttt{pfs}^{\neg}(\mathcal{L})$. The proof goes by induction on complexity. Assume to start that $\texttt{comp}(A) < 2$. Thus either: (1) $A = \mathcal{T}$; (2) $A = \neg\mathcal{T}$; (3) $A = \bot$; (4) $A = \neg\bot$; (5) $A = p_i$; or (6) $A = \neg p_i$. Consider:

Case 1: Assume $A = \mathcal{T}$. Thus $\texttt{neg}(A) = \mathcal{T}$, so $\texttt{gen}(\texttt{neg}(A)) = \mathcal{T}$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Case 2: Assume $A = \neg\mathcal{T}$. Thus $\texttt{neg}(A) = \mathcal{V}$, and so $\texttt{gen}(\texttt{neg}(A)) = \neg\mathcal{T}$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Case 3: Assume $A = \bot$. Thus $\texttt{neg}(A) = \bot$, so $\texttt{gen}(\texttt{neg}(A)) = \bot$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Case 4: Assume $A = \neg\bot$. Thus $\texttt{neg}(A) = \underline{\bot}$, and so $\texttt{gen}(\texttt{neg}(A)) = \neg\bot$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Case 5: Assume $A = p_i$. Thus $\texttt{neg}(A) = p_{2i}$, and so $\texttt{gen}(\texttt{neg}(A)) = A$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Case 6: Assume $A = \neg p_i$. Thus $\texttt{neg}(A) = p_{2i+1}$, and so $\texttt{gen}(\texttt{neg}(A)) = A$. We may then conclude that $\texttt{gen}(\texttt{neg}(A)) = A$.

Since $\texttt{gen}(\texttt{neg}(A)) = A$ holds in each of the cases above, we know by **E1** that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$, thereby completing the base case.

Assume for induction that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ for all $A \in \texttt{pfs}^{\neg}(\mathcal{L})$ such that $\texttt{comp}(A) < n$. Let $A \in \texttt{pfs}^{\neg}(\mathcal{L})$ be such that $\texttt{comp}(A) = n$. It follows that either: (1) $A = \neg\neg B$; (2) $A = B \wedge C$; (3) $A = B \vee C$; (4) $A = \neg(B \vee C)$; or (5) $A = \neg(B \wedge C)$. Consider the following:

Case 1: Assume $A = \neg\neg B$. By definition, $\texttt{neg}(A) = \texttt{neg}(B)$, and so $\texttt{gen}(\texttt{neg}(A)) = \texttt{gen}(\texttt{neg}(B))$ where $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \approx B$ by hypothesis. Thus if follows that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx B$. Since $\vdash_{\text{UGS}} B \approx \neg\neg B$ by **E17**, we may conclude that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx \neg\neg B$ by **GA9**, and so $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ follows by assumption.

Case 2: Assume $A = B \wedge C$. Thus $\texttt{neg}(A) = \texttt{neg}(B) \wedge \texttt{neg}(C)$, and so $\texttt{gen}(\texttt{neg}(A)) = \texttt{gen}(\texttt{neg}(B) \wedge \texttt{neg}(C))$. It follows by definition that $\texttt{gen}(\texttt{neg}(B) \wedge \texttt{neg}(C)) = \texttt{gen}(\texttt{neg}(B)) \wedge \texttt{gen}(\texttt{neg}(C))$, where we know that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \approx B$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(C)) \approx C$ by hypothesis. It follows that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \wedge \texttt{gen}(\texttt{neg}(C)) \approx B \wedge C$ by **GA8**, and so $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ follows from the identities above.

Case 3: Assume $A = B \vee C$. Thus $\texttt{neg}(A) = \texttt{neg}(B) \vee \texttt{neg}(C)$, and $\texttt{gen}(\texttt{neg}(A)) = \texttt{gen}(\texttt{neg}(B) \vee \texttt{neg}(C))$. However, we know by definition that $\texttt{gen}(\texttt{neg}(B) \vee \texttt{neg}(C)) = \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$, where both $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \approx B$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(C)) \approx C$ follow by hypothesis. Additionally, $\vdash_{\text{UGS}} B \trianglelefteq B \vee C$ and $\vdash_{\text{UGS}} C \trianglelefteq B \vee C$ by **GA1** and **GA2**, and so $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \trianglelefteq B \vee C$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(C)) \trianglelefteq B \vee C$ by **GA9**. Thus $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C)) \trianglelefteq B \vee C$ follows by **AR1**. Similarly, we know that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \trianglelefteq \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(C)) \trianglelefteq \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$ by **GA1** and **GA2**, and so it follows that both $\vdash_{\text{UGS}} B \trianglelefteq \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$ and $\vdash_{\text{UGS}} C \trianglelefteq \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$ by **GA9**. Thus we know by **AR1** that $\vdash_{\text{UGS}} B \vee C \trianglelefteq \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C))$, and so given the above, $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \vee \texttt{gen}(\texttt{neg}(C)) \approx B \vee C$. Thus we may conclude by the identities above that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ as desired.

Case 4: Assume $A = \neg(B \vee C)$. Thus $\texttt{neg}(A) = \texttt{neg}(\neg B) \wedge \texttt{neg}(\neg C)$, and so $\texttt{gen}(\texttt{neg}(A)) = \texttt{gen}(\texttt{neg}(\neg B)) \wedge \texttt{gen}(\texttt{neg}(\neg C))$. By hypothesis, $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \approx \neg B$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg C$, and so by **GA8**, $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \wedge \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg B \wedge \neg C$. Additionally, we know that $\vdash_{\text{UGS}} \neg B \wedge \neg C \approx \neg(B \wedge C)$ by **E15**, and so it follows by **GA9** that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \wedge \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg(B \vee C)$. Given the identities above, we may conclude that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$.

Case 5: Assume $A = \neg(B \wedge C)$. Thus $\texttt{neg}(A) = \texttt{neg}(\neg B) \vee \texttt{neg}(\neg C)$, and so $\texttt{gen}(\texttt{neg}(A)) = \texttt{gen}(\texttt{neg}(\neg B) \vee \texttt{neg}(\neg C))$. It follows by definition that $\texttt{gen}(\texttt{neg}(\neg B) \vee \texttt{neg}(\neg C)) = \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$, where $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \approx \neg B$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg C$ by hypothesis. By **GA1** and **GA2**, $\vdash_{\text{UGS}} \neg B \trianglelefteq \neg B \vee \neg C$ and $\vdash_{\text{UGS}} \neg C \trianglelefteq \neg B \vee \neg C$, and so $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \trianglelefteq \neg B \vee \neg C$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg C)) \trianglelefteq \neg B \vee \neg C$ by **GA9**. Thus $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C)) \trianglelefteq \neg B \vee \neg C$ follows by **AR1**. Similarly, $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \trianglelefteq \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg C)) \trianglelefteq \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$ follow by **GA1** and **GA2**, and so by **GA9** $\vdash_{\text{UGS}} \neg B \trianglelefteq \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$ and $\vdash_{\text{UGS}} \neg C \trianglelefteq \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$. Thus we may conclude that $\vdash_{\text{UGS}} \neg B \vee \neg C \trianglelefteq \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C))$ by **AR1**, and so given the above, $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg B \vee \neg C$. However, we also know by **E16** $\vdash_{\text{UGS}} \neg B \vee \neg C \approx \neg(B \wedge C)$, and so again by **GA9** it follows that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\neg B)) \vee \texttt{gen}(\texttt{neg}(\neg C)) \approx \neg(B \wedge C)$. Thus by the identities above $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ as desired.

Thus $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ holds in each of the cases above, and so it follows by induction that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ for all $A \in \texttt{pfs}^{\neg}(\mathcal{L})$. □

L7.11 $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$, for all $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$.

Proof. Assume $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$. The proof goes by induction where we may assume to start that $\texttt{comp}^{+}(\varphi) = 0$. Thus either $\varphi = A \trianglelefteq B$ for some $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, or else $\varphi = \$A$ for some $A \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Case I: Assume $\varphi = A \trianglelefteq B$ for some $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. It follows that $\texttt{gen}(\texttt{neg}(\varphi)) = \texttt{gen}(\texttt{neg}(A)) \trianglelefteq \texttt{gen}(\texttt{neg}(B))$, and so by **L7.10** both $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ and $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(B)) \approx B$. Thus we know by **GA9** that $\vdash_{\text{UGS}} (\texttt{gen}(\texttt{neg}(A)) \trianglelefteq \texttt{gen}(\texttt{neg}(B))) \leftrightarrow (A \trianglelefteq B)$, and so it follows that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$ given the identity above.

Case II: Assume $\varphi = \$A$ for some $A \in \texttt{pfs}^{\neg}(\mathcal{L})$. Thus it follows that $\texttt{gen}(\texttt{neg}(\varphi)) = \$\,\texttt{gen}(\texttt{neg}(A))$, where $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(A)) \approx A$ by **L7.10**. By **SP2**, $\texttt{gen}(\texttt{neg}(A)) \approx A \vdash_{\text{UGS}} \$\,\texttt{gen}(\texttt{neg}(A)) \leftrightarrow \$A$, and so we may conclude that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$.

Since $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$ holds in both of the base cases, we know that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$. We may then conclude by a routine induction proof that $\vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi)) \leftrightarrow \varphi$ for all $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$. □

P7.2 $\Sigma \vdash_{\text{UGS}} \varphi$ iff $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$, for all $\Sigma \cup \{\varphi\} \subseteq \texttt{wfs}^{\neg}(\mathcal{L})$.

Proof. Let $\Sigma \cup \{\varphi\} \subseteq \texttt{wfs}^{\neg}(\mathcal{L})$, and assume $\Sigma \vdash_{\text{UGS}} \varphi$. We argue by induction on the length of proof, where we assume to start that $\Sigma \vdash^{0}_{\text{UGS}} \varphi$. It follows that $\Sigma \vdash_{\text{UGS}} \varphi$ holds by one of the rules or axioms of UG.

Case GA1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **GA1**, and so $\Sigma = \varnothing$ and $\varphi = A \trianglelefteq A \vee B$ for some $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. However, we know by **L7.8** that $\texttt{neg}(A), \texttt{neg}(B) \in \texttt{pfs}(\mathcal{L}^{-})$, and so $\vdash_{\text{UGSN}} \texttt{neg}(A) \trianglelefteq \texttt{neg}(A) \vee \texttt{neg}(B)$ since **GA1** belongs to UGSN. Thus $\vdash_{\text{UGSN}} \texttt{neg}(A \trianglelefteq A \vee B)$ as desired. We may then conclude that $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. I will omit consideration of the other grounding rules and axioms, all of which are similar.

Case SP1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **SP1**, and so $\Sigma = \{\$A\}$ and $\varphi = A \not\trianglelefteq \bot$. We know by **L7.8** that $\texttt{neg}(A) \in \texttt{pfs}(\mathcal{L}^{-})$, and so $\$\,\texttt{neg}(A) \vdash_{\text{UGSN}} \texttt{neg}(A) \not\trianglelefteq \bot$ since **SP1** belongs to UGSN. It follows that $\texttt{neg}(\$A) \vdash_{\text{UGSN}} \texttt{neg}(A \not\trianglelefteq \bot)$, since by definition both $\texttt{neg}(\$A) = \$\,\texttt{neg}(A)$ and $\texttt{neg}(A \not\trianglelefteq \bot) = \texttt{neg}(A) \not\trianglelefteq \bot$. Thus $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. I will omit consideration of **SP2** – **SP5**, all of which are similar to **SP1**.

Case NA1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **NA1**, and so $\Sigma = \varnothing$ and $\varphi = A \trianglelefteq \neg\neg A$ for some $A \in \texttt{pfs}^{\neg}(\mathcal{L})$. By **L7.8**, $\texttt{neg}(A) \in \texttt{pfs}(\mathcal{L}^{-})$, and so $\vdash_{\text{UGSN}} \texttt{neg}(A) \trianglelefteq \texttt{neg}(A)$ follows by **E1**. However, we also know that $\texttt{neg}(A \trianglelefteq \neg\neg A) = \texttt{neg}(A) \trianglelefteq \texttt{neg}(A)$, and so $\vdash_{\text{UGSN}} \texttt{neg}(A \trianglelefteq \neg\neg A)$. Thus $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$ as desired. I will omit consideration of **NA2** – **NA6** which are similar.

Since $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$ holds in each of the cases above, we may conclude that if $\Sigma \vdash^{0}_{\text{UGS}} \varphi$, then $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. Assume for induction that for all $k < n$, if $\Sigma \vdash^{k}_{\text{UGS}} \varphi$, then $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. Assume for discharge that $\Sigma \vdash^{n}_{\text{UGS}} \varphi$. Thus $\Sigma \vdash^{n}_{\text{UGS}} \varphi$ follows from one of the metarules included in UG. Consider the following cases.

Case SP6: Assume $\Sigma \vdash^{n}_{\text{UGS}} \varphi$ follows by **SP6**. Thus $\varphi = A \trianglelefteq B$ where $\Sigma \vdash^{k}_{\text{UGS}} \$p_i \to [(p_i \trianglelefteq A) \to (p_i \trianglelefteq B)]$ for some $k < n$ and $p_i \in \mathbb{L}$ which does not occur in A, B, or in any $\sigma \in \Sigma$. We may then conclude by hypothesis that $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\$p_i \to [(p_i \trianglelefteq A) \to (p_i \trianglelefteq B)])$, and so $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \$p_{2i} \to ([p_{2i} \trianglelefteq \texttt{neg}(A)] \to [p_{2i} \trianglelefteq \texttt{neg}(B)])$. Given that p_i does not occur in A, B, or in any $\sigma \in \Sigma$, we may observe that p_{2i} does not occur in $\texttt{neg}(A)$, $\texttt{neg}(B)$, or in $\texttt{neg}(\sigma)$ for any $\sigma \in \Sigma$. It follows that $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(A) \trianglelefteq \texttt{neg}(B)$ by **SP6**, and so $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(A \trianglelefteq B)$. Thus we may then conclude that $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$ as desired. I will omit consideration of **SP7** which is similar.

Given the cases above, it follows by induction that if $\Sigma \vdash_{\text{UGS}} \varphi$, then $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. In order to prove the converse, assume instead that $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$. It follows that $\texttt{neg}(\Sigma) \vdash_{\text{UGS}} \texttt{neg}(\varphi)$ since UG extends UGSN. By **R2**$^{\pm}$, $\texttt{gen}(\texttt{neg}(\Sigma)) \vdash_{\text{UGS}} \texttt{gen}(\texttt{neg}(\varphi))$, and so $\Sigma \vdash_{\text{UGS}} \varphi$ by **L7.11**. Together with the above, we may conclude that $\Sigma \vdash_{\text{UGS}} \varphi$ just in case $\texttt{neg}(\Sigma) \vdash_{\text{UGSN}} \texttt{neg}(\varphi)$ for all $\Sigma \cup \{\varphi\} \subseteq \texttt{wfs}^{\neg}(\mathcal{L})$. □

We are now in a position to introduce the function $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$, showing every $\mathcal{M} \in \mathcal{C}^{\infty}$ is such that $\mathcal{M} \vDash \texttt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \vDash \varphi$. Consider:

$\mathfrak{N}$-*Model:* Given any $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$, let $|\cdot|^{\mathfrak{N}} = \langle |\cdot|^{\mathfrak{N}^{+}}, |\cdot|^{\mathfrak{N}^{-}} \rangle$ where $|p_i|^{\mathfrak{N}^{+}} = |p_{2i}|$ and $|p_i|^{\mathfrak{N}^{-}} = |p_{2i+1}|$ for all $p_i \in \mathbb{L}$.

$\mathfrak{N}$-Map: Given any $\mathcal{M} \in \mathcal{C}^\infty$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot|\rangle$, let $\mathcal{M}^\mathfrak{N} = \langle S, \bigsqcup, |\cdot|^\mathfrak{N}\rangle$.

We may then prove the following lemmas.

L7.12 $\mathcal{M}^\mathfrak{N} \in \mathcal{C}^\pm$ for all $\mathcal{M} \in \mathcal{C}^\infty$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\infty$. It follows that $\mathcal{M} = \langle S, \bigsqcup, |\cdot|\rangle$ for some $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup\rangle$. By definition, $|p| \in \mathbb{P}^\infty_\mathcal{S}$ for all $p \in \mathbb{L}$. Choose some $p_i \in \mathbb{L}$. It follows that both $|p_{2i}|, |p_{2i+1}| \in \mathbb{P}^\infty_\mathcal{S}$, and so $\langle |p_{2i}|, |p_{2i+1}|\rangle \in \mathbb{P}^\pm_\mathcal{S}$. Thus $|p_i|^\mathfrak{N} \in \mathbb{P}^\infty_\mathcal{S}$ since $|p_i|^\mathfrak{N} = \langle |p_i|^{\mathfrak{N}^+}, |p_i|^{\mathfrak{N}^-}\rangle$ where $|p_i|^{\mathfrak{N}^+} = |p_{2i}|$ and $|p_i|^{\mathfrak{N}^-} = |p_{2i+1}|$. Since $p_i \in \mathbb{L}$ was arbitrary, we may conclude that $\mathcal{M}^\mathfrak{N} \in \mathcal{C}^\pm$ where $\mathcal{M}^\mathfrak{N} = \langle S, \bigsqcup, |\cdot|^\mathfrak{N}\rangle$. □

L7.13 $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ and $\mathcal{M} \in \mathcal{C}^\infty$.

Proof. The proof goes by induction on complexity. Assume to start that $A \in \mathtt{pfs}^\neg(\mathcal{L})$ where $\mathtt{comp}(A) < 2$. Thus either: (1) $A = \mathcal{T}$; (2) $A = \neg\mathcal{T}$; (3) $A = \bot$; (4) $A = \neg\bot$; (5) $A = p_i$; or (6) $A = \neg p_i$. Consider:

Case 1: Let $A = \mathcal{T}$, and so $\mathtt{neg}(A) = \mathcal{T}$. Thus $|\mathtt{neg}(A)| = |\mathcal{T}|$. Since $|\mathcal{V}|^{\mathfrak{N}^+} = |\mathcal{T}|$, we know that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 2: Let $A = \neg\mathcal{T}$, and so $\mathtt{neg}(A) = \mathcal{V}$. Thus it follows that $|\mathtt{neg}(A)| = |\mathcal{V}|$. By **L7.12** and **L7.1**, $|\neg\mathcal{T}|^{\mathfrak{N}^+} = |\mathcal{T}|^{\mathfrak{N}^-}$, where $|\mathcal{T}|^{\mathfrak{N}^-} = |\mathcal{V}|$, and so $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 3: Let $A = \bot$, and so $\mathtt{neg}(A) = \bot$. Thus $|\mathtt{neg}(A)| = |\bot|$. Since $|\bot|^{\mathfrak{N}^+} = |\bot|$, we know that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 4: Let $A = \neg\bot$, and so $\mathtt{neg}(A) = \perp\!\!\!\!-$. Thus it follows that $|\mathtt{neg}(A)| = |\perp\!\!\!\!-|$. By **L7.12** and **L7.1**, $|\neg\bot|^{\mathfrak{N}^+} = |\bot|^{\mathfrak{N}^-}$, where $|\bot|^{\mathfrak{N}^-} = |\perp\!\!\!\!-|$, and so $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 5: Let $A = p_i$ for some $p_i \in \mathbb{L}$. It follows that $\mathtt{neg}(A) = p_{2i}$, and so $|\mathtt{neg}(A)| = |p_{2i}|$. We also know that $|p_i|^{\mathfrak{N}^+} = |p_{2i}|$, and so $|A|^{\mathfrak{N}^+} = |p_{2i}|$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 6: Let $A = \neg p_i$ for some $p_i \in \mathbb{L}$. Thus $\mathtt{neg}(A) = p_{2i+1}$, and so it follows that $|\mathtt{neg}(A)| = |p_{2i+1}|$. By **L7.12** and **L7.1**, $|\neg p_i|^{\mathfrak{N}^+} = |p_i|^{\mathfrak{N}^-}$, where $|p_i|^{\mathfrak{N}^-} = |p_{2i+1}|$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Given the cases above, $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ where $\mathtt{comp}(A) < 2$. Assume for induction that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ such that $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}^\neg(\mathcal{L})$ be such that $\mathtt{comp}(A) = n$. Thus it follows that either: (*a*) $A = \neg\neg B$; (*b*) $A = B \wedge C$; (*c*) $A = B \vee C$; (*d*) $A = \neg(B \vee C)$; or (*e*) $A = \neg(B \wedge C)$.

Case (*a*)*:* Assume $A = \neg\neg B$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B)$, where we know that $|\mathtt{neg}(B)| = |B|^{\mathfrak{N}^+}$ by hypothesis. By **L7.12** and **L7.1**, it follows that $|B|^{\mathfrak{N}^+} = |\neg B|^{\mathfrak{N}^-} = |\neg\neg B|^{\mathfrak{N}^+}$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case (b): Assume $A = B \wedge C$. Thus $\texttt{neg}(A) = \texttt{neg}(B) \wedge \texttt{neg}(C)$, where both $|\texttt{neg}(B)| = |B|^{\mathfrak{N}+}$ and $|\texttt{neg}(C)| = |C|^{\mathfrak{N}+}$ by hypothesis. By **L7.12** and **L7.2**, $|B \wedge C|^{\mathfrak{N}+} = |B|^{\mathfrak{N}+} \wedge |C|^{\mathfrak{N}+}$. It follows that:

$$\begin{aligned} |\texttt{neg}(A)| &= |\texttt{neg}(B) \wedge \texttt{neg}(C)| \\ (*) &= |\texttt{neg}(B)| \wedge |\texttt{neg}(C)| \\ &= |B|^{\mathfrak{N}+} \wedge |C|^{\mathfrak{N}+} \\ &= |B \wedge C|^{\mathfrak{N}+} \\ &= |A|^{\mathfrak{N}+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^{+}_{\wedge}]$ given in **L7.2**, where the other identities follow from the above. Thus $|\texttt{neg}(A)| = |A|^{\mathfrak{N}+}$.

Case (c): Assume $A = B \vee C$. Thus $\texttt{neg}(A) = \texttt{neg}(B) \vee \texttt{neg}(C)$, where both $|\texttt{neg}(B)| = |B|^{\mathfrak{N}+}$ and $|\texttt{neg}(C)| = |C|^{\mathfrak{N}+}$ by hypothesis. By **L7.12** and **L7.3**, $|B \vee C|^{\mathfrak{N}+} = |B|^{\mathfrak{N}+} \vee |C|^{\mathfrak{N}+}$. It follows that:

$$\begin{aligned} |\texttt{neg}(A)| &= |\texttt{neg}(B) \vee \texttt{neg}(C)| \\ (*) &= |\texttt{neg}(B)| \vee |\texttt{neg}(C)| \\ &= |B|^{\mathfrak{N}+} \vee |C|^{\mathfrak{N}+} \\ &= |B \vee C|^{\mathfrak{N}+} \\ &= |A|^{\mathfrak{N}+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^{+}_{\vee}]$ given in **L7.3**, where the other identities follow from the above. Thus $|\texttt{neg}(A)| = |A|^{\mathfrak{N}+}$.

Case (d): Assume $A = \neg(B \vee C)$, so $\texttt{neg}(A) = \texttt{neg}(\neg B) \wedge \texttt{neg}(\neg C)$. By hypothesis, both $|\texttt{neg}(\neg B)| = |\neg B|^{\mathfrak{N}+}$ and $|\texttt{neg}(\neg C)| = |\neg C|^{\mathfrak{N}+}$. By **L7.12** and **L7.1**, $|\neg B|^{\mathfrak{N}+} = |B|^{\mathfrak{N}-}$ and $|\neg C|^{\mathfrak{N}+} = |C|^{\mathfrak{N}-}$, as well as $|\neg(B \vee C)|^{\mathfrak{N}+} = |B \vee C|^{\mathfrak{N}-}$, where $|B \vee C|^{\mathfrak{N}-} = |B|^{\mathfrak{N}-} \wedge |C|^{\mathfrak{N}-}$ follows by **L7.3**. We may then argue as follows:

$$\begin{aligned} |\texttt{neg}(A)| &= |\texttt{neg}(\neg B) \wedge \texttt{neg}(\neg C)| \\ (*) &= |\texttt{neg}(\neg B)| \wedge |\texttt{neg}(\neg C)| \\ &= |\neg B|^{\mathfrak{N}+} \wedge |\neg C|^{\mathfrak{N}+} \\ &= |B|^{\mathfrak{N}-} \wedge |C|^{\mathfrak{N}-} \\ &= |B \vee C|^{\mathfrak{N}-} \\ &= |\neg(B \vee C)|^{\mathfrak{N}+} \\ &= |A|^{\mathfrak{N}+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^{+}_{\wedge}]$ given in **L7.2**, where the other identities follow from the above. Thus $|\texttt{neg}(A)| = |A|^{\mathfrak{N}+}$.

Case (e): Assume $A = \neg(B \wedge C)$, so $\texttt{neg}(A) = \texttt{neg}(\neg B) \vee \texttt{neg}(\neg C)$. By hypothesis, $|\texttt{neg}(\neg B)| = |\neg B|^{\mathfrak{N}+}$ and $|\texttt{neg}(\neg C)| = |\neg C|^{\mathfrak{N}+}$. By **L7.12** and **L7.1**, we know that $|\neg B|^{\mathfrak{N}+} = |B|^{\mathfrak{N}-}$ and $|\neg C|^{\mathfrak{N}+} = |C|^{\mathfrak{N}-}$, as well as $|\neg(B \wedge C)|^{\mathfrak{N}+} = |B \wedge C|^{\mathfrak{N}-}$, where $|B \wedge C|^{\mathfrak{N}-} = |B|^{\mathfrak{N}-} \vee |C|^{\mathfrak{N}-}$

follows by **L7.2**. We may then argue as follows:

$$\begin{aligned} |\,\texttt{neg}(A)| &= |\,\texttt{neg}(\neg B) \vee \texttt{neg}(\neg C)| \\ (*) &= |\,\texttt{neg}(\neg B)| \vee |\,\texttt{neg}(\neg C)| \\ &= |\neg B|^{\mathfrak{N}^+} \vee |\neg C|^{\mathfrak{N}^+} \\ &= |B|^{\mathfrak{N}^-} \vee |C|^{\mathfrak{N}^-} \\ &= |B \wedge C|^{\mathfrak{N}^-} \\ &= |\neg(B \wedge C)|^{\mathfrak{N}^+} \\ &= |A|^{\mathfrak{N}^+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^+_\vee]$ given in **L7.3**, where the other identities follow from the above. Thus $|\,\texttt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Given that $|\,\texttt{neg}(A)| = |A|^{\mathfrak{N}^+}$ in each of the cases above, we may conclude by induction that $|\,\texttt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \texttt{pfs}^\neg(\mathcal{L})$. □

P7.3 $\mathcal{M} \models \texttt{neg}(\varphi)$ iff $\mathcal{M}^{\mathfrak{N}} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}^\infty$ and $\varphi \in \texttt{wfs}^\neg(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\infty$ and $\varphi \in \texttt{wfs}^\neg(\mathcal{L})$. The proof goes by induction on complexity. Assume $\texttt{comp}^+(\varphi) = 0$. It follows that either $\varphi = A \trianglelefteq B$ for some $A, B \in \texttt{pfs}^\neg(\mathcal{L})$, or $\varphi = \$A$ for some $A \in \texttt{pfs}^\neg(\mathcal{L})$.

Case I: Assume $\varphi = A \trianglelefteq B$ for some $A, B \in \texttt{pfs}^\neg(\mathcal{L})$. It follows that $\texttt{neg}(\varphi) = \texttt{neg}(A) \trianglelefteq \texttt{neg}(B)$. We may then reason as follows:

$$\begin{aligned} \mathcal{M} \models \texttt{neg}(\varphi) \ &\textit{iff} \ \mathcal{M} \models \texttt{neg}(A) \trianglelefteq \texttt{neg}(B) \\ &\textit{iff} \ |\,\texttt{neg}(A)| \subseteq |\,\texttt{neg}(B)| \\ (*) \ &\textit{iff} \ |A|^{\mathfrak{N}+} \subseteq |B|^{\mathfrak{N}+} \\ &\textit{iff} \ \mathcal{M}^{\mathfrak{N}} \models A \trianglelefteq B \\ &\textit{iff} \ \mathcal{M}^{\mathfrak{N}} \models \varphi. \end{aligned}$$

The biconditionals above all follow by definition or assumption with the exception of $(*)$ which is given by **L7.13**. Thus we may conclude that $\mathcal{M} \models \texttt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$ as desired.

Case II: Assume $\varphi = \$A$ for some $A \in \texttt{pfs}^\neg(\mathcal{L})$. It follows that $\texttt{neg}(\varphi) = \$\,\texttt{neg}(A)$. We may then reason as follows:

$$\begin{aligned} \mathcal{M} \models \texttt{neg}(\varphi) \ &\textit{iff} \ \mathcal{M} \models \$\,\texttt{neg}(A) \\ &\textit{iff} \ |\,\texttt{neg}(A)| = \{s\} \text{ for some } s \in S \\ (*) \ &\textit{iff} \ |A|^{\mathfrak{N}+} = \{s\} \text{ for some } s \in S \\ &\textit{iff} \ \mathcal{M}^{\mathfrak{N}} \models \$A \\ &\textit{iff} \ \mathcal{M}^{\mathfrak{N}} \models \varphi. \end{aligned}$$

As before, the biconditionals above all follow by definition or assumption with the exception of $(*)$ which is given by **L7.13**. Thus we may conclude that $\mathcal{M} \models \texttt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$.

Given the cases above, it follows by a routine induction proof that $\mathcal{M} \models \texttt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$ for any $\varphi \in \texttt{wfs}^\neg(\mathcal{L})$. □

P7.4 $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$ is a surjection.

Proof. Choose some $\mathcal{M}_u \in \mathcal{C}^{\pm}_{\mathcal{S}}$. It follows that $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u \rangle$, where $|p_i|_u = \langle |p_i|^+_u, |p_i|^-_u \rangle$ for all $p_i \in \mathbb{L}$. We may then define $|\cdot|$ as follows:

$$|p_i| = \begin{cases} |p_{\frac{i}{2}}|^+_u & \text{if } i \text{ is even} \\ |p_{\frac{i-1}{2}}|^- & \text{otherwise.} \end{cases}$$

Since $\mathcal{M}_u \in \mathcal{C}^{\pm}$, we know that $\langle |p_i|^+_u, |p_i|^-_u \rangle \in \mathbb{P}^{\pm}_{\mathcal{S}}$, and so $|p_i|^+_u, |p_i|^-_u \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $|p_{2i}|, |p_{2i+1}| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $i \in \mathbb{N}$, and so $|p_i| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $p_i \in \mathbb{L}$. By definition, $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle \in \mathcal{C}^{\infty}$. We may then consider the following:

$$\begin{aligned} |p_i|^{\mathfrak{N}} &= \langle |p_i|^{\mathfrak{N}^+}, |p_i|^{\mathfrak{N}^-} \rangle \\ &= \langle |p_{2i}|, |p_{2i+1}| \rangle \\ &= \langle |p_i|^+_u, |p_i|^-_u \rangle \\ &= |p_i|_u. \end{aligned}$$

The identities above all hold by definition or assumption. It follows that $\mathcal{M}^{\mathfrak{N}} = \mathcal{M}_u$, and so there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M}^{\mathfrak{N}} = \mathcal{M}_u$ for any $\mathcal{M}_u \in \mathcal{C}^{\pm}$. Thus we may conclude that $\mathfrak{N}$ is surjective. □

8 Bilattice Theory

Having extended *Soundness* and *Completeness* to UGS, we may now employ the resources above to define bilateral analogues of $\propto$ and $\subseteq$ along with a unary inversion operator over the space of propositions $\mathbb{P}^{\pm}_{\mathcal{S}}$, showing that the resulting structure forms a bilattice. To begin with, consider the following orders:

Bilateral Containment: $\langle X, Y \rangle \sqsubseteq \langle U, V \rangle$ *iff* $X \propto U$ and $Y \subseteq V$.

Bilateral Entailment: $\langle X, Y \rangle \Subset \langle U, V \rangle$ *iff* $X \subseteq U$ and $Y \propto V$.

In order to prove that both $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset \rangle$ and $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \sqsubseteq \rangle$ are complete lattices, we may show that for any $\mathcal{S} \in \mathbb{M}_{\infty}$ and indexed family of propositions $\{P_i : i \in I\} \subseteq \mathbb{P}^{\pm}_{\mathcal{S}}$, $\bigwedge\{P_i : i \in I\}$ is the least upper bound with respect to $\sqsubseteq$, and $\bigvee\{P_i : i \in I\}$ is the least upper bound with respect to $\Subset$ as follows:

L8.1 $\bigwedge\{P_i : i \in I\} = \texttt{lub}^{\sqsubseteq}\{P_i : i \in I\}$.

L8.2 $\bigvee\{P_i : i \in I\} = \texttt{lub}^{\Subset}\{P_i : i \in I\}$.

Since both $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset \rangle$ and $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \sqsubseteq \rangle$ are complete lattices, it follows that $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq \rangle$ is a *pre-bilattice*. Accordingly, $\mathcal{B}^{\pm}_{\mathcal{S}} = \langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq, \neg \rangle$ is a *bilattice* on account of the fact that $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq \rangle$ is a pre-bilattice where '$\neg$' is a unary operator on $\mathbb{P}^{\pm}_{\mathcal{S}}$ which obeys the following conditions for all $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.3 $\neg\neg P = P$.

L8.4 If $P \Subset Q$, then $\neg P \sqSubset \neg Q$.

L8.5 If $P \sqSubset Q$, then $\neg P \Subset \neg Q$.

It remains to further characterise the properties which $\mathcal{B}^{\pm}_{\mathcal{S}}$ exhibits, marking important points of departure from Boolean theories in which the space of propositions forms a complemented distributive lattice.[18]

Given any $\mathcal{S} \in \mathcal{C}^{\pm}$, we may begin by showing that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is bounded below by proving that the following identities hold for any $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.6 $P \vee \bot = P$.

L8.7 $P \wedge \Bot = P$.

Observe that $\bigvee \varnothing = \bot$ and $\bigwedge \varnothing = \Bot$, and so $\bigvee \varnothing$ and $\bigwedge \varnothing$ provide lower bounds for $\mathbb{P}^{\pm}_{\mathcal{S}}$ with respect to $\Subset$ and $\sqSubset$. By contrast, $\bigvee \mathbb{P}^{\pm}_{\mathcal{S}} = \langle S, \varnothing \rangle \neq \mathcal{T}$ and $\bigwedge \mathbb{P}^{\pm}_{\mathcal{S}} = \langle \varnothing, S \rangle \neq \mathcal{V}$ given that $\bot, \Bot \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Rather we find that $\bigvee \mathbb{P}^{\pm}_{\mathcal{S}} = \mathcal{T} \vee \Bot$ and $\bigwedge \mathbb{P}^{\pm}_{\mathcal{S}} = \bot \wedge \mathcal{V}$. Letting $\top = \mathcal{T} \vee \Bot$ and $\Top = \bot \wedge \mathcal{V}$, it follows that:

L8.8 $P \vee \top = \top$.

L8.9 $P \wedge \Top = \Top$.

It follows that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is bounded above by $\top$ and $\Top$ with respect to $\Subset$ and $\sqSubset$ rather than $\mathcal{T}$ and $\mathcal{V}$.[19] Nevertheless, $\top$ and $\Top$ may be defined in terms of $\mathcal{T}, \bot, \mathcal{V}$, and $\Bot$, whereas the same cannot be said in reverse.[20]

We may now turn to observe that both $\wedge$ and $\vee$ are monotonic over their respective orders. More specifically, for any $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.13 If $P \sqSubset Q$, then $P \wedge R \sqSubset Q \wedge R$.

L8.14 If $P \Subset Q$, then $P \vee R \Subset Q \vee R$.

By contrast with the above, $\wedge$ and $\vee$ are not monotonic over each other's orders given that there are some $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for which:

L8.15 $P \sqSubset Q$ and $P \vee R \not\sqSubset Q \vee R$.

L8.16 $P \Subset Q$ and $P \wedge R \not\Subset Q \wedge R$.

It follows that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is not *interlaced* which requires $\wedge$ and $\vee$ to be monotonic over both orders $\sqSubset$ and $\Subset$.[21] We may also show that both absorption laws fail, where $\mathcal{B}^{\pm}_{\mathcal{S}}$ is distributive only in negation since conjunction does not distribute over disjunction, nor does disjunction distribute over conjunction:

---

[18] Ginsberg (1988) first proposed the definition of a bilattice. Alternatively, one could drop the requirement that pre-bilattices consist of lattices which are *complete* as in Rivieccio (2010). See also Mobasher et al. (2000) for discussion of such discrepancies in usage.

[19] See §7 of Chapter 4 and Conclusion for related discussion.

[20] However, by letting $\mathbb{P}^{(\pm)}_{\mathcal{S}} = \{\langle V, F \rangle : V, F \in \mathbb{P}^{\infty}_{\mathcal{S}}/\varnothing\}$, we may show that both $\bigwedge(\mathbb{P}^{(\pm)}_{\mathcal{S}}) = \mathcal{V}$ and $\bigvee(\mathbb{P}^{(\pm)}_{\mathcal{S}}) = \mathcal{T}$. See Fine (2017b, p. 642) for related discussion.

[21] Restricting $\mathbb{P}^{\infty}_{\mathcal{S}}$ to convex propositions makes the resulting pre-bilattice interlaced.

L8.17 $P \wedge (P \vee Q) \neq P$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

L8.18 $P \vee (P \wedge Q) \neq P$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

L8.19 $\neg(P \wedge Q) = \neg P \vee \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}_\mathcal{S}$.

L8.20 $\neg(P \vee Q) = \neg P \wedge \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}_\mathcal{S}$.

L8.21 $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

L8.22 $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

The symmetry in distribution laws for conjunction over disjunction and *vice versa* may be contrasted with the asymmetry between **L3.21** and **L3.22**. Having offered a preliminary survey of the structure of $\mathbb{P}^\pm_\mathcal{S}$, I will turn to relate these results to the theorems of UGS in the following section.

L8.1 $\bigwedge\{P_i : i \in I\} = \texttt{lub}^{\sqsubseteq}\{P_i : i \in I\}$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^\pm_\mathcal{S}$, where $P_i = \langle P_i^+, P_i^- \rangle$ for all $i \in I$. By definition, $\bigwedge\{P_i : i \in I\} = \langle \bigwedge\{P_i^+ : i \in I\}, \bigvee\{P_i^- : i \in I\}\rangle$. Since $P_i^\pm \in \mathbb{P}^\infty_\mathcal{S}$ for all $i \in I$, both $\bigwedge\{P_i^+ : i \in I\} = \texttt{lub}^{\trianglerighteq}\{P_i^+ : i \in I\}$ and $\bigvee\{P_i^- : i \in I\} = \texttt{lub}^{\subseteq}\{P_i^- : i \in I\}$ follow from **L6.6** and **L6.7**, respectively. Thus $P_i^+ \trianglerighteq \bigwedge\{P_i^+ : i \in I\}$ and $P_i^- \subseteq \bigvee\{P_i^- : i \in I\}$ for all $i \in I$, and so $P_i \sqsubseteq \bigwedge\{P_i : i \in I\}$ for all $i \in I$. It follows that $\bigwedge\{P_i : i \in I\}$ is an upper bound of $\{P_i : i \in I\}$ with respect to $\sqsubseteq$.

Let $Z \in \mathbb{P}^\pm_\mathcal{S}$, and assume that $P_i \sqsubseteq Z$ for all $i \in I$. Thus $Z = \langle Z^+, Z^- \rangle$ and $P_i = \langle P_i^+, P_i^- \rangle$ for each $i \in I$, where $Z^\pm, P_i^\pm \in \mathbb{P}^\infty_\mathcal{S}$. Again by **L6.6** and **L6.7**, we know that $\bigwedge\{P_i^+ : i \in I\} \trianglerighteq Z^+$ and $\bigvee\{P_i^- : i \in I\} \subseteq Z^-$, and so $\bigwedge\{P_i : i \in I\} \sqsubseteq Z$. Since Z was arbitrary, we may conclude from the above that $\bigwedge\{P_i : i \in I\} = \texttt{lub}^{\sqsubseteq}\{P_i : i \in I\}$. □

L8.2 $\bigvee\{P_i : i \in I\} = \texttt{lub}^{\Subset}\{P_i : i \in I\}$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.1**. □

L8.3 $\neg\neg P = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^\pm \in \mathbb{P}^\infty_\mathcal{S}$. By definition, $\neg\langle P^+, P^- \rangle = \langle P^-, P^+ \rangle$, and so it follows that $\neg\neg\langle P^+, P^- \rangle = \langle P^+, P^- \rangle$. Equivalently, $\neg\neg P = P$. □

L8.4 For all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$, if $P \Subset Q$, then $\neg P \sqsubseteq \neg Q$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$, and assume $P \Subset Q$. Thus it follows that $P = \langle P^+, P^- \rangle$ and $Q = \langle Q^+, Q^- \rangle$ for some $P^\pm, Q^\pm \in \mathbb{P}^\infty_\mathcal{S}$, where both $P^+ \subseteq Q^+$ and $P^- \trianglerighteq Q-$. Thus $\langle P^-, P^+ \rangle \sqsubseteq \langle Q^-, Q^+ \rangle$, and so $\neg\langle P^+, P^- \rangle \sqsubseteq \neg\langle Q^+, Q^- \rangle$. Equivalently, $\neg P \sqsubseteq \neg Q$ as desired. □

L8.5 For all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$, if $P \sqsubseteq Q$, then $\neg P \Subset \neg Q$.

Proof. Similar to **L8.4**. □

L8.6 $P \vee \bot = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Since both $P^- \vee \varnothing = P^-$ and $P^+ \wedge \{\square\} = P^+$, it follows that $\langle P^+ \wedge \{\square\}, P^- \vee \varnothing \rangle = \langle P^+, P^- \rangle$, and so $P \wedge \bot = P$ by definition. ☐

L8.7 $P \wedge \underline{\bot} = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Similar to **L8.6**. ☐

L8.8 $P \vee \top = \top$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and so $P^+ \vee S = S$ by **L6.12**, and $P^- \wedge \varnothing = \varnothing$ by **L6.13**. Thus $\langle P^+ \vee S, P^- \wedge \varnothing \rangle = \langle S, \varnothing \rangle$, and so $P \vee \top = \top$ as desired. ☐

L8.9 $P \wedge \overline{\top} = \overline{\top}$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Similar to **L8.8**. ☐

L8.10 $\mathcal{S}^3 \in \mathbb{M}_\infty$ where $\mathcal{S}^3 = \langle \mathcal{P}(\{a, b, c\}), \bigcup \rangle$.

Proof. Let $\mathcal{S}^3 = \langle \mathcal{P}(\{a, b, c\}), \bigcup \rangle$. Since $\bigcup$ satisfies *Associativity* where $\bigcup\{x\} = x$ for all $x \in \mathcal{P}(\{a, b, c\})$, we know that $\mathcal{S}^3 \in \mathbb{M}_\infty$. ☐

L8.11 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$, if $X \trianglerighteq Y$, then $X \wedge Z \trianglerighteq Y \wedge Z$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and assume $X \trianglerighteq Y$. It follows that $X \gg Y$ and $X \ll Y$. Letting $s \in Y \wedge Z$, we know that $s = \bigsqcup\{x, y\}$ for some $x \in Y$ and $y \in Z$, and so there is some $z \in X$ where $z \sqsubseteq x$. Thus $\bigsqcup\{z, y\} \in Y \wedge Z$ where $\bigsqcup\{z, x\} = x$. We may then reason as follows:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{z, y\}, s\} &= \bigsqcup\{\bigsqcup\{z, y\}, \bigsqcup\{x, y\}\} \\
&= \bigsqcup\bigcup\{\{z, y\}, \{x, y\}\} \\
&= \bigsqcup\bigcup\{\{z, x\}, \{y\}\} \\
&= \bigsqcup\{\bigsqcup\{z, x\}, \bigsqcup\{y\}\} \\
&= \bigsqcup\{x, y\} \\
&= s.
\end{aligned}$$

The identities stated above follow by *Collapse* and *Associativity* given our assumptions. Thus $\bigsqcup\{\bigsqcup\{z, y\}, s\} = s$, and so $\bigsqcup\{z, y\} \sqsubseteq s$. Generalising on s, we may conclude that that $X \wedge Z \gg Y \wedge Z$.

Choose instead some $u \in X \wedge Z$ and $v \in Y \wedge Z$. Thus $u = \bigsqcup\{x, z\}$ and $v = \bigsqcup\{y, w\}$ for some $x \in X$, $y \in Y$, and $z, w \in Z$. Since $X \ll Y$, we know that $\bigsqcup\{x, y\} \in Y$, where $\bigsqcup\{z, w\} \in Z$ given that $Z \in \mathbb{P}_{\mathcal{S}}$. Accordingly, $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{z, w\}\} \in Y \wedge Z$, and so $\bigsqcup\{\bigsqcup\{x, z\}, \bigsqcup\{y, w\}\} \in Y \wedge Z$, so $\bigsqcup\{u, v\} \in Y \wedge Z$. Thus $X \wedge Z \ll Y \wedge Z$, and so $X \wedge Z \trianglerighteq Y \wedge Z$. ☐

L8.12 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$, if $X \subseteq Y$, then $X \vee Z \subseteq Y \vee Z$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$, and assume $X \trianglerighteq Y$. Choose $s \in X \vee R^+$. By **L6.8**, $s \in X \cup Z \cup (X \wedge Z)$. If $s \in X \cup Z$, then $s \in Y \cup Z$, and so $s \in Y \cup Z \cup (Y \wedge Z)$. If $s \in X \wedge Z$, then $s = \bigsqcup\{x, y\}$ for some $x \in X$ and $y \in Z$, and so $s \in Y \wedge Z$ since $x \in Y$. Thus $s \in Y \cup Z \cup (Y \wedge Z)$, and so $s \in Y \vee Z$ by **L6.8**. It follows that $X \vee Z \subseteq Y \vee Z$. □

L8.13 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$, if $P \sqsubseteq Q$, then $P \wedge R \sqsubseteq Q \wedge R$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$, and assume $P \sqsubseteq Q$ for discharge. Thus $P^+ \trianglerighteq Q^+$ and $P^- \subseteq Q^-$, where $P^\pm, Q^\pm, R^\pm \in \mathbb{P}^\infty_\mathcal{S}$. It follows that $P^+ \wedge R^+ \trianglerighteq Q^+ \wedge R^+$ by **L8.11**, and $P^- \vee R^- \subseteq Q^- \vee R^-$ by **L8.12**. Accordingly, $\langle P^+ \wedge R^+, P^- \vee R^- \rangle \sqsubseteq \langle Q^+ \wedge R^+, Q^- \vee R^- \rangle$, and so by definition we may conclude that $P \wedge R \sqsubseteq Q \wedge R$. □

L8.14 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$, if $P \Subset Q$, then $P \vee R \Subset Q \vee R$.

Proof. Similar to **L8.13**. □

L8.15 $P \sqsubseteq Q$ and $P \vee R \not\sqsubseteq Q \vee R$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. By **L3.12**, we know that $\mathcal{S}_3 \in \mathbb{M}_\infty$. Let $P = \langle \{\{a\}\}, \{\{a\}\} \rangle$, $Q = \langle \{\{a\}, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\} \rangle$, and $R = \langle \{\{c\}\}, \{\{c\}\} \rangle$, and observe that $P, Q, R \in \mathbb{P}^\pm_{\mathcal{S}_3}$. For the sake of readability, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. We may then observe the following:

$$P = \langle \{\alpha\}, \{\alpha\} \rangle$$
$$Q = \langle \{\alpha.\beta\}, \{\alpha.\beta\} \rangle$$
$$R = \langle \{\gamma\}, \{\gamma\} \rangle$$

$$P \vee R = \langle \{\alpha, \gamma, \alpha.\gamma\}, \{\alpha.\gamma\} \rangle$$
$$Q \vee R = \langle \{\beta, \alpha.\beta, \alpha.\beta.\gamma\}, \{\alpha.\beta.\gamma\} \rangle.$$

Given that $\gamma \in P \vee R$ and $\beta \in Q \vee R$, but $\gamma.\beta \notin Q \vee R$, we may conclude that $P \vee R \not\sqsubseteq Q \vee R$. However, $P \sqsubseteq Q$, concluding the proof. □

L8.16 $P \Subset Q$ and $P \wedge R \not\Subset Q \wedge R$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.15**. □

L8.17 $P \neq P \wedge (P \vee Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. By **L3.12**, we know that $\mathcal{S}_3 \in \mathbb{M}_\infty$. Let $P = \langle \{\{a\}\}, \{\{a\}\} \rangle$ and $Q = \langle \{\{b\}\}, \{\{b\}\} \rangle$, observing that $P, Q \in \mathbb{P}^\pm_{\mathcal{S}_3}$. For ease of exposition, let $\alpha = \{a\}$ and $\beta = \{b\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. Thus it follows that:

$$P \vee Q = \langle \{\alpha, \beta, \alpha.\beta\}, \{\alpha.\beta\} \rangle$$
$$P \wedge (P \vee Q) = \langle \{\alpha, \alpha.\beta\}, \{\alpha, \alpha.\beta\} \rangle.$$

We may conclude the proof by observing that $P \neq P \wedge (P \vee Q)$. □

L8.18 $P \neq P \vee (P \wedge Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

Proof. Similar to **L8.17**. □

L8.19 $\neg(P \wedge Q) = \neg P \vee \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows that $P = \langle P^+, P^- \rangle$ and $Q = \langle Q^+, Q^- \rangle$ for some $P^{\pm}, Q^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. We may then argue as follows:

$$\begin{aligned} \neg(P \wedge Q) &= \neg(\langle P^+, P^- \rangle \wedge \langle Q^+, Q^- \rangle) \\ &= \neg(\langle P^+ \wedge Q^+, P^- \vee Q^- \rangle) \\ &= \langle P^- \vee Q^-, P^+ \wedge Q^+ \rangle \\ &= \langle P^-, P^+ \rangle \vee \langle Q^-, Q^+ \rangle \\ &= \neg P \vee \neg Q. \end{aligned}$$

The identities above all follow by definition or assumption. □

L8.20 $\neg(P \vee Q) = \neg P \wedge \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Similar to **L8.19**. □

L8.21 $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $P = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $Q = \langle \{\{b\}\}, \{\{c\}\} \rangle$, and $R = \langle \{\{c\}\}, \{\{a\}\} \rangle$. For ease of exposition, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. We may then observe the following:

$$\begin{aligned} P &= \langle \{\alpha\}, \{\beta\} \rangle & Q \wedge R &= \langle \{\alpha.\gamma\}, \{\alpha, \beta, \alpha.\beta\} \rangle \\ Q &= \langle \{\beta\}, \{\gamma\} \rangle & P \vee Q &= \langle \{\alpha, \beta, \alpha.\beta\}, \{\beta.\gamma\} \rangle \\ R &= \langle \{\gamma\}, \{\alpha\} \rangle & P \vee R &= \langle \{\alpha, \gamma, \alpha.\gamma\}, \{\alpha.\gamma\} \rangle. \end{aligned}$$

$$\begin{aligned} P \vee (Q \wedge R) &= \langle \{\alpha, \alpha.\gamma\}, \{\beta, \alpha.\beta\} \rangle \\ (P \vee Q) \wedge (P \vee R) &= \langle \{\alpha, \alpha.\beta, \alpha.\gamma, \alpha.\beta.\gamma\}, \{\alpha.\gamma, \beta.\gamma, \alpha.\beta.\gamma\} \rangle. \end{aligned}$$

By inspection, we may conclude that $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$, observing that $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}^3}$ and $s^3 \in \mathbb{M}_\infty$ by **L8.10**. □

L8.22 $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $P = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $Q = \langle \{\{b\}\}, \{\{c\}\} \rangle$, and $R = \langle \{\{c\}\}, \{\{a\}\} \rangle$. For ease of exposition, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. Given the identities in **L8.21**, it follows that:

$$\begin{aligned} P \wedge Q &= \langle \{\beta.\gamma\}, \{\alpha, \beta, \alpha.\beta\} \rangle \\ P \wedge R &= \langle \{\alpha.\gamma\}, \{\alpha, \gamma, \alpha.\gamma\} \rangle \\ P \wedge (Q \vee R) &= \langle \{\alpha.\gamma\}, \{\alpha, \beta, \alpha.\beta\} \rangle \\ (P \wedge Q) \vee (P \wedge R) &= \langle \{\alpha.\beta, \alpha.\gamma, \alpha.\beta.\gamma\}, \{\beta, \alpha.\beta, \alpha.\gamma, \beta.\gamma, \alpha.\beta.\gamma\} \rangle. \end{aligned}$$

By inspection, we may conclude that $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$, observing that $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}^3}$ and $s^3 \in \mathbb{M}_\infty$ by **L8.10**. □

9 The Logic of Essence and Ground

In order to make the connections between the propositional bilattice and UGS more perspicuous, it will help to introduce a number of further abbreviations, defining essence and ground in terms of unilateral ground as in §1.

Unilateral Essence: Let $\ulcorner A \trianglerighteq B \urcorner$ abbreviate $\ulcorner A \wedge B \approx B \urcorner$.

Ground: Let ⌜$A \leqslant B$⌝ abbreviate ⌜$(A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B)$⌝.

Essence: Let ⌜$A \sqsubseteq B$⌝ abbreviate ⌜$(A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B)$⌝.

Reduction: Let ⌜$A \Rightarrow B$⌝ abbreviate ⌜$(A \leqslant B) \wedge (A \sqsubseteq B)$⌝.

Identity: Let ⌜$A \equiv B$⌝ abbreviate ⌜$(A \Rightarrow B) \wedge (B \Rightarrow A)$⌝.

Whereas unilateral ground only describes the relationship between the exact verifiers for the antecedent and consequent, essence and ground constrain both the exact verifiers and falsifiers for the propositions involved.

The following results will help to bring out the manner in which the operators defined above are able to describe the propositional bilattice $\mathcal{B}^{\pm}_{\mathcal{S}}$ characterised above for any given $\mathcal{S} \in \mathbb{M}_{\infty}$. In particular, I will show that the following hold for any $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$:

L9.1 $\mathcal{M} \vDash A \trianglerighteq B$ *iff* $|A|^{+} \trianglerighteq |B|^{+}$.

L9.2 $\mathcal{M} \vDash A \leqslant B$ *iff* $|A| \Subset |B|$.

L9.3 $\mathcal{M} \vDash A \sqsubseteq B$ *iff* $|A| \sqsubseteq |B|$.

L9.4 $\mathcal{M} \vDash A \Rightarrow B$ *iff* $|A| \Subset |B|$ and $|A| \sqsubseteq |B|$.

L9.5 $\mathcal{M} \vDash A \equiv B$ *iff* $|A| = |B|$.

Given these connections, we may show that each of the following Boolean identities admit of countermodels in the present setting:

L9.6 $\nvDash_{\mathcal{C}^{\pm}} A \equiv A \wedge (A \vee B)$.

L9.7 $\nvDash_{\mathcal{C}^{\pm}} A \equiv A \vee (A \wedge B)$.

L9.8 $\nvDash_{\mathcal{C}^{\pm}} A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

L9.9 $\nvDash_{\mathcal{C}^{\pm}} A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

These results help to distinguish UGS from extensional and intensional logics which describe spaces of propositions which form complemented distributive lattices. Additionally, we may show that analogues of the identities above hold when identity has been replaced with reduction, highlighting the distinctive role which reduction has to play in the present theory.

Rather than continuing to work over UGS, it will help to further exhibit the relationships between essence and ground and the extensional operators by considering the following range of theorems and admissible rules within UGS:

T1 $\bot \leqslant A$.

T3 $\top \sqsubseteq A$.

T5 $A \leqslant A \vee B$.

T7 $A \sqsubseteq A \wedge B$.

T9 $A \wedge B \leqslant A \vee B$.

T11 $A \Rightarrow A \wedge (A \vee B)$.

T13 $A \Rightarrow A \vee (A \wedge B)$.

T2 $A \leqslant \top$.

T4 $A \sqsubseteq \overline{\top}$.

T6 $B \leqslant A \vee B$.

T8 $B \sqsubseteq A \wedge B$.

T10 $A \vee B \sqsubseteq A \wedge B$.

T12 $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

T14 $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$.

R1 $A \leqslant B \ \vdash \ B \sqsubseteq A \vee B$.

R3 $A \sqsubseteq B \ \vdash \ B \leqslant A \wedge B$.

R5 $A \leqslant C,\ B \leqslant C \ \vdash \ A \vee B \leqslant C$.

R7 $A \sqsubseteq C,\ B \sqsubseteq C \ \vdash A \wedge B \sqsubseteq C$.

R9 $A \leqslant B,\ B \leqslant C \ \vdash \ A \leqslant C$.

R11 $A \sqsubseteq B,\ B \sqsubseteq C \ \vdash A \sqsubseteq C$.

R2 $A \leqslant B \ \vdash \ A \vee B \sqsubseteq B$.

R4 $A \sqsubseteq B \ \vdash \ A \wedge B \leqslant B$.

R6 $A \leqslant B,\ A \leqslant C \ \vdash \ A \leqslant B \wedge C$.

R8 $A \sqsubseteq B,\ A \sqsubseteq C \ \vdash A \sqsubseteq B \vee C$.

R10 $A \sqsubseteq B,\ C \sqsubseteq D \ \vdash A \wedge C \sqsubseteq B \wedge D$.

R12 $A \leqslant B,\ C \leqslant D \ \vdash A \vee C \leqslant B \vee D$.

Let the syntactic consequent relation $\vdash_{\text{EG}}$ for *The Logic of Essence and Ground* (EG) be the smallest relation closed under truth-functional consequence which satisfies the above. In contrast to UGS, the subsystem EG aims to capture a natural means of reasoning about essence and ground.

Rather than deriving each of the axioms and rules for EG within UGS, I will provide derivations of **T11** – **T14** as characteristic examples:

L9.10 $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B)$.

L9.11 $\vdash_{\text{UGS}} A \Rightarrow A \vee (A \wedge B)$.

L9.19 $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

L9.20 $\vdash_{\text{UGS}} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$.

In order to further characterise the granularity of the present theory of bilateral propositions, we may also derive the following with UGS, where I will establish the theoremhood of **ID9** and **ID11** by way of **T4** for convenience:

ID1 $A \vee \bot \equiv A$.

ID3 $A \vee \top \equiv \top$.

ID5 $A \equiv A \wedge A$

ID7 $A \wedge B \equiv B \wedge A$

ID9 $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$

ID11 $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

ID13 $A \equiv \neg\neg A$

ID2 $A \wedge \underline{\bot} \equiv A$.

ID4 $A \wedge \top \equiv \top$.

ID6 $A \equiv A \vee A$

ID8 $A \vee B \equiv B \vee A$

ID9 $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

ID11 $(A \vee B) \vee C \equiv A \vee (B \vee C)$

IDR1 $A \equiv B \vdash B \equiv A$

IDR2 $A \equiv B \vdash (A \wedge C) \equiv (B \wedge C)$

IDR4 $A \equiv B,\ B \equiv C \vdash A \equiv C$

IDR3 $A \equiv B \vdash (A \vee C) \equiv (B \vee C)$

IDR5 $A \equiv B \vdash \neg A \equiv \neg B$

Given the results above, we may let the syntactic consequent relation $\vdash_{\text{LPI}}$ for *The Logic of Propositional Identity* (LPI) be the smallest relation closed under truth-functional consequence which satisfies the above. In contrast EG which streamlines reasoning about essence and ground, LPI aims to capture a natural means of reasoning about propositional identity in the present setting. These results help to further characterise the structure of the present theory of bilateral propositions in contrast to the traditional Boolean theories of propositions developed in extensional and intensional logics.

Despite having proven that UGS is sound and complete over $\mathcal{C}^{\pm}$, it remains to show whether the subsystems EG and LPI also admit of completeness results. In particular, one might aim to show that UGS is a conservative extension of both EG as LPI. Leaving these further pursuits for another day, I will conclude by presenting proofs of a selection of the claims asserted above.

L9.1 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \trianglerighteq B$ *iff* $|A|^+ \trianglerighteq |B|^+$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \trianglerighteq B \quad & \textit{iff} \quad \mathcal{M} \vDash B \approx A \wedge B \\ & \textit{iff} \quad |B|^+ = |A \wedge B|^+ \\ (\dagger) \quad & \textit{iff} \quad |B|^+ = |A|^+ \wedge |B|^+ \\ (\ddagger) \quad & \textit{iff} \quad |A|^+ \trianglerighteq |B|^+. \end{aligned}$$

The biconditionals above hold by definition with the exception of (†) which is given by **L7.2**, and (‡) which requires further argument.

Assume $|B|^+ = |A|^+ \wedge |B|^+$, and let $s \in |B|^+$. Thus $s \in |A|^+ \wedge |B|^+$, and so $s = \bigsqcup\{x, y\}$ for some $x \in |A|^+$ and $y \in |B|^+$. It follows that $x \sqsubseteq s$, and so $|A|^+ \gg |B|^+$ since s was arbitrary. Assuming that $x \in |A|^+$ and $y \in |B|^+$, it follows that $\bigsqcup\{x, y\} \in |A|^+ \wedge |B|^+$, and so $\bigsqcup\{x, y\} \in |B|^+$ by assumption. Thus $|A|^+ \ll |B|^+$, and so $|A|^+ \trianglerighteq |B|^+$.

Assume instead that $|A|^+ \trianglerighteq |B|^+$. Thus $|A|^+ \gg |B|^+$ and $|A|^+ \ll |B|^+$. Letting $s \in |B|^+$, it follows that there is some $y \in |A|^+$ where $y \sqsubseteq s$, and so $\bigsqcup\{y, s\} = s$. By definition, $\bigsqcup\{y, s\} \in |A|^+ \wedge |B|^+$, and so $s \in |A|^+ \wedge |B|^+$. Thus $|B|^+ \subseteq |A|^+ \wedge |B|^+$. Assume instead that $t \in |A|^+ \wedge |B|^+$. It follows that $t = \bigsqcup\{x, y\}$ for some $x \in |A|^+$ and $y \in |B|^+$. Since $|A|^+ \ll |B|^+$, it follows that $t \in |B|^+$, and so $|A|^+ \wedge |B|^+ \subseteq |B|^+$. Given the above, $|B|^+ = |A|^+ \wedge |B|^+$, thereby establishing (‡). □

L9.2 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \leqslant B$ *iff* $|A| \Subset |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \leqslant B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B) \\ & \textit{iff} \quad \mathcal{M} \vDash A \trianglelefteq B \text{ and } \mathcal{M} \vDash \neg A \trianglerighteq \neg B \\ (*) \quad & \textit{iff} \quad |A|^{+} \subseteq |B|^{+} \text{ and } |\neg A|^{+} \trianglerighteq |\neg B|^{+} \\ & \textit{iff} \quad |A|^{+} \subseteq |B|^{+} \text{ and } |A|^{-} \trianglerighteq |B|^{-} \\ & \textit{iff} \quad |A| \Subset |B|. \end{aligned}$$

All of the biconditionals above hold by definition with the exception of $(*)$ which follows from the **Bilateral Semantics** together with **L9.1**. □

L9.3 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \sqsubseteq B$ *iff* $|A| \sqsubseteq |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \sqsubseteq B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B) \\ & \textit{iff} \quad \mathcal{M} \vDash A \trianglerighteq B \text{ and } \mathcal{M} \vDash \neg A \trianglelefteq \neg B \\ (*) \quad & \textit{iff} \quad |A|^{+} \trianglerighteq |B|^{+} \text{ and } |\neg A|^{+} \subseteq |\neg B|^{+} \\ & \textit{iff} \quad |A|^{+} \trianglerighteq |B|^{+} \text{ and } |A|^{-} \subseteq |B|^{-} \\ & \textit{iff} \quad |A| \sqsubseteq |B|. \end{aligned}$$

All of the biconditionals above hold by definition with the exception of $(*)$ which follows from the **Bilateral Semantics** together with **L9.1**. □

L9.4 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \Rightarrow B$ *iff* $|A| \Subset |B|$ and $|A| \sqsubseteq |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \Rightarrow B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \leqslant B) \wedge (A \sqsubseteq B) \\ & \textit{iff} \quad \mathcal{M} \vDash A \leqslant B \text{ and } \mathcal{M} \vDash A \sqsubseteq B \\ (*) \quad & \textit{iff} \quad |A| \Subset |B| \text{ and } |A| \sqsubseteq |B|. \end{aligned}$$

The biconditionals above follow by definition with the exception of $(*)$ which is given by **L9.2** and **L9.3**. □

L9.5 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \equiv B$ *iff* $|A| = |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \equiv B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \Rightarrow B) \wedge (B \Rightarrow A) \\ & \textit{iff} \quad \mathcal{M} \vDash A \Rightarrow B \text{ and } \mathcal{M} \vDash B \Rightarrow A \\ (\dagger) \quad & \textit{iff} \quad |A| \Subset |B|, |A| \sqsubseteq |B|, |B| \Subset |A|, \text{ and } |B| \sqsubseteq |A| \\ (\ddagger) \quad & \textit{iff} \quad |A| = |B|. \end{aligned}$$

The biconditionals above all follow by definition with the exception of (†) which is given by **L9.4**, and (‡) which requires further support.

Assume that: (1) $|A| \Subset |B|$; (2) $|A| \sqsubseteq |B|$; (3) $|B| \Subset |A|$; and (4) $|B| \sqsubseteq |A|$. It follows that $|A|^{+} \subseteq |B|^{+}$ and $|B|^{+} \subseteq |A|^{+}$ from (1) and (3), where similarly $|A|^{-} \subseteq |B|^{-}$ and $|B|^{-} \subseteq |A|^{-}$ follow from (2) and (4). Thus we may conclude that $|A| = |B|$.

Assume instead that $|A| = |B|$. Thus $|A|^{\pm} = |B|^{\pm}$ and $|B|^{\pm} = |A|^{\pm}$, and so $|A|^{\pm} \subseteq |B|^{\pm}$ and $|B|^{\pm} \subseteq |A|^{\pm}$. Since $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$, both $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **P7.1**, and so $|A|^{\pm}, |B|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $|A|^{\pm} \trianglerighteq |B|^{\pm}$ and $|B|^{\pm} \trianglerighteq |A|^{\pm}$ follow by **L9.14**, and so (1) – (4) follow by definition. □

L9.6 $\not\models_{\mathcal{C}^{\pm}} A \equiv A \wedge (A \vee B)$ for some $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Recall that $P \neq P \wedge (P \vee Q)$ for some $\mathcal{S} \in \mathbb{M}_{\infty}$ and $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **L8.17**. Let $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ where $|p_1| = P$ and $|p_2| = Q$. It follows that $|p_1| \neq |p_1| \wedge (|p_1| \vee |p_2|)$. Thus we may reason as follows:

$$\begin{aligned} |p_1| \neq |p_1| \wedge (|p_1| \vee |p_2|) &\quad \textit{iff} \quad |p_1| \neq |p_1| \wedge |p_1 \vee p_2| \\ &\quad \textit{iff} \quad |p_1| \neq |p_1 \wedge (p_1 \vee p_2)| \\ (*) &\quad \textit{iff} \quad \mathcal{M} \not\models p_1 \equiv p_1 \wedge (p_1 \vee p_2). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \equiv [p_1 \wedge (p_2 \vee p_3)]$. □

L9.7 $\not\models_{\mathcal{C}^{\pm}} A \equiv A \vee (A \wedge B)$ for some $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Recall that $P \neq P \vee (P \wedge Q)$ for some $\mathcal{S} \in \mathbb{M}_{\infty}$ and $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **L8.18**. Let $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ where $|p_1| = P$ and $|p_2| = Q$. It follows that $|p_1| \neq |p_1| \vee (|p_1| \wedge |p_2|)$. Thus we may reason as follows:

$$\begin{aligned} |p_1| \neq |p_1| \vee (|p_1| \wedge |p_2|) &\quad \textit{iff} \quad |p_1| \neq |p_1| \vee |p_1 \wedge p_2| \\ &\quad \textit{iff} \quad |p_1| \neq |p_1 \vee (p_1 \wedge p_2)| \\ (*) &\quad \textit{iff} \quad \mathcal{M} \not\models p_1 \equiv p_1 \vee (p_1 \wedge p_2). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \equiv [p_1 \vee (p_2 \wedge p_3)]$. □

L9.8 $\not\models_{\mathcal{C}^{\pm}} A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ for some $A, B, C \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Recall that $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_{\infty}$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **L8.21**. Let $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ where $|p_1| = P$, $|p_2| = Q$ and $|p_3| = R$. Thus $|p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|)$, and so:

$$\begin{aligned} &|p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|) \\ &\quad \textit{iff} \quad |p_1| \vee |p_2 \wedge p_3| \neq |p_1 \vee p_2| \wedge |p_1 \vee p_3| \\ &\quad \textit{iff} \quad |p_1 \vee (p_2 \wedge p_3)| \neq |(p_1 \vee p_2) \wedge (p_1 \vee p_3)| \\ (*) &\quad \textit{iff} \quad \mathcal{M} \not\models p_1 \vee (p_2 \wedge p_3) \equiv (p_1 \vee p_2) \wedge (p_1 \vee p_3). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \vee (p_2 \wedge p_3) \equiv (p_1 \vee p_2) \wedge (p_1 \vee p_3)$. □

L9.9 $\not\models_{\mathcal{C}^{\pm}} A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ for some $A, B, C \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Recall that $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_{\infty}$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **L8.22**. Let $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ where $|p_1| = P$, $|p_2| = Q$, and $|p_3| = R$. It follows that $|p_1| \wedge (|p_2| \vee |p_3|) \neq (|p_1| \wedge |p_2|) \vee (|p_1| \wedge |p_3|)$. Thus we may reason as follows:

$$\begin{aligned} &|p_1| \wedge (|p_2| \vee |p_3|) \neq (|p_1| \wedge |p_2|) \vee (|p_1| \wedge |p_3|) \\ &\quad \textit{iff} \quad |p_1| \wedge |p_2 \vee p_3| \neq |p_1 \wedge p_2| \vee |p_1 \wedge p_3| \\ &\quad \textit{iff} \quad |p_1 \wedge (p_2 \vee p_3)| \neq |(p_1 \wedge p_2) \vee (p_1 \wedge p_3)| \\ (*) &\quad \textit{iff} \quad \mathcal{M} \not\models p_1 \wedge (p_2 \vee p_3) \equiv (p_1 \wedge p_2) \vee (p_1 \wedge p_3). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \wedge (p_2 \vee p_3) \equiv (p_1 \wedge p_2) \vee (p_1 \vee p_3)$. □

L9.10 $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B)$ for all $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. We know that $\vdash_{\text{UGS}} A \wedge [A \vee (A \wedge B)] \trianglelefteq (A \wedge A) \vee [A \wedge (A \wedge B)]$ by **E13**, where $\vdash_{\text{UGS}} A \wedge A \trianglelefteq A \vee (A \wedge B)$ follows from **GA4** and **GA1**. Additionally, $\vdash_{\text{UGS}} A \wedge (A \wedge B) \trianglelefteq A \vee (A \wedge B)$ holds by **GA5**, **GA4**, **GA8**, and **GA2**. Thus $\vdash_{\text{UGS}} (A \wedge A) \vee [A \wedge (A \wedge B)] \trianglelefteq A \vee (A \wedge B)$ follows by **AR1**, and so $\vdash_{\text{UGS}} A \wedge [A \vee (A \wedge B)] \trianglelefteq A \vee (A \wedge B)$ holds by **GA9**. Next observe that $\vdash_{\text{UGS}} A \wedge (A \wedge B) \trianglelefteq A \wedge [A \vee (A \wedge B)]$ follows from **GA2**, **E1**, and **GA8**, and so $\vdash_{\text{UGS}} A \wedge B \trianglelefteq A \wedge [A \vee (A \wedge B)]$ holds by **GA6**, **GA3**, **E1**, and **GA8**. Additionally, $\vdash_{\text{UGS}} A \trianglelefteq A \wedge [A \vee (A \wedge B)]$ follows from **E1**, **GA1**, **GA8**, **GA3**, and **GA9**, and so we may conclude that $\vdash_{\text{UGS}} A \vee (A \wedge B) \trianglelefteq A \wedge [A \vee (A \wedge B)]$ by **AR1**. Together with the above, it follows that $\vdash_{\text{UGS}} A \vee (A \wedge B) \approx A \wedge [A \vee (A \wedge B)]$, and so by $\mathbf{R2}^{\pm}$ $\vdash_{\text{UGS}} \neg A \vee (\neg A \wedge \neg B) \approx \neg A \wedge [\neg A \vee (\neg A \wedge \neg B)]$. By **NE12**, **NE13**, and **AR4**, it follows that $\vdash_{\text{UGS}} \neg A \vee (\neg A \wedge \neg B) \approx \neg A \wedge \neg[A \wedge (A \vee B)]$, and so $\vdash_{\text{UGS}} \neg A \trianglerighteq \neg[A \wedge (A \vee B)]$ by definition.

Given that $\vdash_{\text{UGS}} A \trianglelefteq A \wedge (A \vee B)$ by **T3**, it follows from the above that $\vdash_{\text{UGS}} A \leqslant A \wedge (A \vee B)$. Since we may show by a similar argument that $\vdash_{\text{UGS}} A \sqsubseteq A \wedge (A \vee B)$, we may conclude that $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B)$. □

L9.11 $\vdash_{\text{UGS}} A \Rightarrow A \vee (A \wedge B)$ for all $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Similar to **L9.10**. □

L9.12 $\vdash_{\text{UGS}} \neg A \vee \neg B \equiv \neg(A \wedge B)$ for all $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$. By **P7.1**, $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$, and so $\neg(|A| \wedge |B|) = \neg|A| \vee \neg|B|$ by **L8.19**. Consider the following:

$$\begin{aligned} \neg(|A| \wedge |B|) = \neg|A| \vee \neg|B| \quad & \textit{iff} \quad \neg|A \wedge B| = |\neg A| \wedge |\neg B| \\ & \textit{iff} \quad |\neg(A \wedge B)| = |\neg A \vee \neg B| \\ (*) \quad & \textit{iff} \quad \vDash_{\mathcal{C}^{\pm}} \neg(A \wedge B) \equiv \neg A \vee \neg B. \end{aligned}$$

The biconditionals above follow by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\vDash_{\mathcal{C}^{\pm}} \neg(A \vee B) \equiv \neg A \wedge \neg B$, and so we may conclude that $\vdash_{\text{UGS}} \neg(A \vee B) \equiv \neg A \wedge \neg B$ by **Theorem T4**. □

L9.13 $\vDash_{\mathcal{C}^{\pm}} \neg(A \vee B) \equiv \neg A \wedge \neg B$ for all $A, B \in \texttt{pfs}^{\neg}(\mathcal{L})$.

Proof. Similar to **L9.10**, drawing on **L8.20** in place of **L8.19**. □

L9.14 For all $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y \in \mathbb{P}^{\infty}_{\mathcal{S}}$, if $X \subseteq Y$, then $Y \trianglerighteq X$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and assume $X \subseteq Y$. Choose some $s \in X$. It follows that $s \in Y$, where $s \sqsubseteq s$ since $\bigsqcup\{s, s\} = s$ by *Collapse*. Generalising on s, we may conclude that $Y \trianglerighteq X$. □

L9.15 $X \wedge (Y \vee Z) \subseteq (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y, Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y, Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and assume $s \in X \wedge (Y \vee Z)$. Thus $s = \bigsqcup\{x, u\}$ for some $x \in X$ and $u \in Y \vee Z$. By **L6.8**, $u \in Y \cup Z \cup (Y \wedge Z)$. If $u \in Y \cup Z$, then $s \in (X \wedge Y) \cup (X \wedge Z)$, and so $s \in (X \wedge Y) \vee (X \wedge Z)$ by **L6.8**. Assume instead $u \in Y \wedge Z$. It follows that $u = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$, and so $\bigsqcup\{x, y\} \in X \wedge Y$ and $\bigsqcup\{x, z\} \in X \wedge Z$. Thus $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} \in (X \wedge Y) \wedge (X \wedge Z)$. Observe the following:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} &= \bigsqcup\bigcup\{\{x, y\}, \{x, z\}\} \\
&= \bigsqcup\bigcup\{\{x\}, \{y, z\}\} \\
&= \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\{y, z\}\} \\
&= \bigsqcup\{x, u\} \\
&= s.
\end{aligned}$$

Thus $s \in (X \wedge Y) \wedge (X \wedge Z)$, and so $s \in (X \wedge Y) \vee (X \wedge Z)$ by **L6.8**. Given the above, $X \wedge (Y \vee Z) \subseteq (X \wedge Y) \vee (X \wedge Z)$ as needed. □

L9.16 $X \vee (Y \wedge Z) \subseteq (X \vee Y) \wedge (X \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$, and assume $s \in X \vee (Y \wedge Z)$. By **L6.8**, $s \in X \cup (Y \wedge Z) \cup [X \wedge (Y \wedge Z)]$. If $s \in X$, then $s \in X \vee Y$ and $s \in X \vee Z$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. Assume instead that $s \in Y \wedge Z$, then $s = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$. By **L6.8**, $y \in X \vee Y$ and $z \in X \vee Z$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. If instead $s \in X \wedge (Y \wedge Z)$, then $s = \bigsqcup\{x, u\}$ for some $x \in X$ and $u \in Y \wedge Z$, and so $u = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$. By **L6.8**, $\bigsqcup\{x, y\} \in X \vee Y$ and $\bigsqcup\{x, z\} \in X \vee Z$, and so $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} \in (X \vee Y) \wedge (X \vee Z)$. By the same argument given in **L9.15**, $s = \bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\}$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. Thus we may conclude that $X \vee (Y \wedge Z) \subseteq (X \vee Y) \wedge (X \vee Z)$. □

L9.17 $X \wedge (Y \vee Z) \trianglerighteq (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Immediate from **L9.15** and **L9.14**. □

L9.18 $X \vee (Y \wedge Z) \trianglerighteq (X \vee Y) \wedge (X \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Immediate from **L9.16** and **L9.14**. □

L9.19 $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ for all $A, B, C \in \texttt{pfs}^\neg(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\pm$ and $A, B, C \in \texttt{pfs}^\neg(\mathcal{L})$. Thus $|A|, |B|, |C| \in \mathbb{P}^\pm_\mathcal{S}$ by **P7.1**, and so $|A|^\pm, |B|^\pm, |C|^\pm \in \mathbb{P}^\infty_\mathcal{S}$. Consider the following:

$$\begin{aligned}
&\mathcal{M} \vDash A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C) \text{ and} \\
&\qquad\quad \mathcal{M} \vDash A \wedge (B \vee C) \sqsubseteq (A \wedge B) \vee (A \wedge C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \wedge (B \vee C) \trianglelefteq (A \wedge B) \vee (A \wedge C), && (1) \\
&\qquad\quad \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglerighteq \neg[(A \wedge B) \vee (A \wedge C)], && (2) \\
&\qquad\quad \mathcal{M} \vDash A \wedge (B \vee C) \trianglerighteq (A \wedge B) \vee (A \wedge C), \text{ and} && (3) \\
&\qquad\quad \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglelefteq \neg[(A \wedge B) \vee (A \wedge C)]. && (4)
\end{aligned}$$

Whereas (1) follows immediately from **T6** by **Theorem T4**, each of the remaining conjuncts require further argument.

Conjunct 2: Since $|A|^- \wedge (|B|^- \vee |C|^-) \trianglerighteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-)$ by **L9.17**, we may argue as follows:

$$\begin{aligned}
&|A|^- \wedge (|B|^- \vee |C|^-) \trianglerighteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-)\\
&\quad \textit{iff}\ \ |A|^- \wedge |B \vee C|^- \trianglerighteq |A \wedge B|^- \vee |A \wedge C|^-\\
&\quad \textit{iff}\ \ |A \wedge (B \vee C)|^- \trianglerighteq |(A \wedge B) \vee (A \wedge C)|^-\\
&\quad \textit{iff}\ \ |\neg[A \wedge (B \vee C)]|^+ \trianglerighteq |\neg[(A \wedge B) \vee (A \wedge C)]|^+\\
&\quad \textit{iff}\ \ \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglerighteq \neg[(A \wedge B) \vee (A \wedge C)].
\end{aligned}$$

The biconditionals above follow from **L7.1** – **L7.3** with the exception of the last which follows from **L9.1**. This establishes (2) as desired.

Conjunct 3: Since $|A|^+ \wedge (|B|^+ \vee |C|^+) \trianglerighteq (|A|^+ \wedge |B|^+) \vee (|A|^+ \wedge |C|^+)$ by **L9.17**, we may argue as follows:

$$\begin{aligned}
&|A|^+ \wedge (|B|^+ \vee |C|^+) \trianglerighteq (|A|^+ \wedge |B|^+) \vee (|A|^+ \wedge |C|^+)\\
&\quad \textit{iff}\ \ |A|^+ \wedge |B \vee C|^+ \trianglerighteq |A \wedge B|^+ \vee |A \wedge C|^+\\
&\quad \textit{iff}\ \ |A \wedge (B \vee C)|^+ \trianglerighteq |(A \wedge B) \vee (A \wedge C)|^+\\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \wedge (B \vee C) \trianglerighteq (A \wedge B) \vee (A \wedge C).
\end{aligned}$$

The biconditionals above follow from **L7.2** and **L7.3** with the exception of the last which follows from **L9.1**. This establishes (3) as desired.

Conjunct 4: By **T6**, $\vdash_{\text{UGS}} \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$, and so $\vDash_{\mathcal{C}^\pm} \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$ by **Theorem T4**. Thus $\mathcal{M} \vDash \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$, and so:

$$\begin{aligned}
&\mathcal{M} \vDash \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)\\
&\quad \textit{iff}\ \ |\neg A \wedge (\neg B \vee \neg C)|^+ \subseteq |(\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)|^+\\
&\quad \textit{iff}\ \ |\neg A|^+ \wedge |\neg B \vee \neg C|^+ \subseteq |\neg A \wedge \neg B|^+ \vee |\neg A \wedge \neg C|^+\\
&\quad \textit{iff}\ \ |\neg A|^+ \wedge (|\neg B|^+ \vee |\neg C|^+) \subseteq (|\neg A|^+ \wedge |\neg B|^+) \vee (|\neg A|^+ \wedge |\neg C|^+)\\
&\quad \textit{iff}\ \ |A|^- \wedge (|B|^- \vee |C|^-) \subseteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-)\\
&\quad \textit{iff}\ \ |A|^- \wedge |B \vee C|^- \subseteq |A \wedge B|^- \vee |A \wedge C|^-\\
&\quad \textit{iff}\ \ |A \wedge (B \vee C)|^- \subseteq |(A \wedge B) \vee (A \wedge C)|^-\\
&\quad \textit{iff}\ \ |\neg[A \wedge (B \vee C)]|^+ \subseteq |\neg[(A \wedge B) \vee (A \wedge C)]|^+\\
&\quad \textit{iff}\ \ \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglelefteq \neg[(A \wedge B) \vee (A \wedge C)].
\end{aligned}$$

The biconditionals above follow from **L7.1** – **L7.3** with the exception of the last which follows by the **Unilateral Semantics**. This proves (4).

Since all of the conditions (1) – (4) are met, we may conclude that $\mathcal{M} \vDash A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$, and so by generalising on $\mathcal{M} \in \mathcal{C}^\pm$, we know that $\vDash_{\mathcal{C}^\pm} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$. Thus it follows by **Theorem T4** that $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$. □

L9.20 $\vDash_{\mathcal{C}^\pm} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$ for all $A, B, C \in \texttt{pfs}^\neg(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\pm$ and $A, B, C \in \texttt{pfs}^\neg(\mathcal{L})$. Thus $|A|, |B|, |C| \in \mathbb{P}^\pm_{\mathcal{S}}$ by

P7.1, and so $|A|^{\pm}, |B|^{\pm}, |C|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following:

$$\begin{aligned}
&\mathcal{M} \vDash A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C) \text{ and} \\
&\quad\quad\ \ \mathcal{M} \vDash A \vee (B \wedge C) \sqsubseteq (A \vee B) \wedge (A \vee C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C), && (1) \\
&\quad\quad\ \ \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglerighteq \neg[(A \vee B) \wedge (A \vee C)], && (2) \\
&\quad\quad\ \ \mathcal{M} \vDash A \vee (B \wedge C) \trianglerighteq (A \vee B) \wedge (A \vee C), \text{ and} && (3) \\
&\quad\quad\ \ \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglelefteq \neg[(A \vee B) \wedge (A \vee C)]. && (4)
\end{aligned}$$

Whereas (1) follows immediately from **T6** by **Theorem T4**, each of the remaining conjuncts require further argument.

Conjunct 2: As $|A|^- \vee (|B|^- \wedge |C|^-) \trianglerighteq (|A|^- \vee |B|^-) \wedge (|A|^- \vee |C|^-)$ by **L9.18**, we may provide the following argument:

$$\begin{aligned}
&|A|^- \vee (|B|^- \wedge |C|^-) \trianglerighteq (|A|^- \vee |B|^-) \wedge (|A|^- \vee |C|^-) \\
&\quad \textit{iff}\ \ |A|^- \vee |B \wedge C|^- \trianglerighteq |A \vee B|^- \wedge |A \vee C|^- \\
&\quad \textit{iff}\ \ |A \vee (B \wedge C)|^- \trianglerighteq |(A \vee B) \wedge (A \vee C)|^- \\
&\quad \textit{iff}\ \ |\neg[A \vee (B \wedge C)]|^+ \trianglerighteq |\neg[(A \vee B) \wedge (A \vee C)]|^+ \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglerighteq \neg[(A \vee B) \wedge (A \vee C)].
\end{aligned}$$

Each of the biconditionals above hold by definition with the exception of the last which follows from **L9.1**. This establishes (2) as desired.

Conjunct 3: As $|A|^+ \vee (|B|^+ \wedge |C|^+) \trianglerighteq (|A|^+ \vee |B|^+) \wedge (|A|^+ \vee |C|^+)$ by **L9.18**, we may provide the following argument:

$$\begin{aligned}
&|A|^+ \vee (|B|^+ \wedge |C|^+) \trianglerighteq (|A|^+ \vee |B|^+) \wedge (|A|^+ \vee |C|^+) \\
&\quad \textit{iff}\ \ |A|^+ \vee |B \wedge C|^+ \trianglerighteq |A \vee B|^+ \wedge |A \vee C|^+ \\
&\quad \textit{iff}\ \ |A \vee (B \wedge C)|^+ \trianglerighteq |(A \vee B) \wedge (A \vee C)|^+ \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \vee (B \wedge C) \trianglerighteq (A \vee B) \wedge (A \vee C).
\end{aligned}$$

Each of the biconditionals above hold by definition with the exception of the last which follows from **L9.1**. This establishes (3) as desired.

Conjunct 4: By **T6**, $\vdash_{\text{UGS}} \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$, and so $\vDash_{\mathcal{C}^{\pm}} \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$ by **Theorem T4**. Thus it follows that $\mathcal{M} \vDash \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$, and so we may reason as follows:

$$\begin{aligned}
&\mathcal{M} \vDash \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \\
&\quad \textit{iff}\ \ |\neg A \vee (\neg B \wedge \neg C)|^+ \subseteq |(\neg A \vee \neg B) \wedge (\neg A \vee \neg C)|^+ \\
&\quad \textit{iff}\ \ |\neg A|^+ \vee |\neg B \wedge \neg C|^+ \subseteq |\neg A \vee \neg B|^+ \wedge |\neg A \vee \neg C|^+ \\
&\quad \textit{iff}\ \ |\neg A|^+ \vee (|\neg B|^+ \wedge |\neg C|^+) \subseteq (|\neg A|^+ \vee |\neg B|^+) \wedge (|\neg A|^+ \vee |\neg C|^+) \\
&\quad \textit{iff}\ \ |A|^- \vee (|B|^- \wedge |C|^-) \subseteq (|A|^- \vee |B|^-) \wedge (|A|^- \vee |C|^-) \\
&\quad \textit{iff}\ \ |A|^- \vee |B \wedge C|^- \subseteq |A \vee B|^- \wedge |A \vee C|^- \\
&\quad \textit{iff}\ \ |A \vee (B \wedge C)|^- \subseteq |(A \vee B) \wedge (A \vee C)|^- \\
&\quad \textit{iff}\ \ |\neg[A \vee (B \wedge C)]|^+ \subseteq |\neg[(A \vee B) \wedge (A \vee C)]|^+ \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglelefteq \neg[(A \vee B) \wedge (A \vee C)].
\end{aligned}$$

The biconditionals above all hold by definition with the exception of the last which follows by the **Unilateral Semantics**, thereby proving (4).

Since all of the conditions (1) – (4) are met, we may conclude that $\mathcal{M} \vDash A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$, and so by generalising on $\mathcal{M} \in \mathcal{C}^{\pm}$, we know that $\vDash_{\mathcal{C}^{\pm}} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$. Thus it follows by **Theorem T4** that $\vdash_{\text{UGS}} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$. ☐

References

Arieli, Ofer and Avron, Arnon. 1996. "Reasoning with Logical Bilattices." *Journal of Logic, Language and Information* 5:25–63.

Fine, Kit. 2001. "The Question of Realism." *Philosophers' Imprint* 1:1–30.

—. 2012a. "Counterfactuals Without Possible Worlds." *Journal of Philosophy* 109:221–246. ISSN 0022-362X. doi: 10.5840/jphil201210938.

—. 2012b. "Guide to Ground." In Fabrice Correia and Benjamin Schnieder (eds.), *Metaphysical Grounding: Understanding the Structure of Reality*, 37–80. Cambridge: Cambridge University Press.

—. 2012c. "The Pure Logic of Ground." *The Review of Symbolic Logic* 5:1–25. ISSN 1755-0211. doi: 10.1017/S1755020311000086.

—. 2013. "Truth-Maker Semantics for Intuitionistic Logic." *Journal of Philosophical Logic* 43:549–577. ISSN 0022-3611, 1573-0433. doi: 10.1007/s10992-013-9281-7.

—. 2014. "Permission and Possible Worlds." *Dialectica* 68:317–336. ISSN 1746-8361. doi: 10.1111/1746-8361.12068.

—. 2015. "Unified Foundations for Essence and Ground." *Journal of the American Philosophical Association* 1:296–311. ISSN 2053-4477. doi: 10.1017/apa.2014.26.

—. 2016. "Angellic Content." *Journal of Philosophical Logic* 45:199–226. ISSN 0022-3611, 1573-0433. doi: 10.1007/s10992-015-9371-9.

—. 2017a. "Naive Metaphysics." *Philosophical Issues* 27:98–113. ISSN 1758-2237. doi: 10.1111/phis.12092.

—. 2017b. "A Theory of Truthmaker Content I: Conjunction, Disjunction and Negation." *Journal of Philosophical Logic* 46:625–674. ISSN 0022-3611, 1573-0433. doi: 10.1007/s10992-016-9413-y.

—. 2017c. "A Theory of Truthmaker Content II: Subject-Matter, Common Content, Remainder and Ground." *Journal of Philosophical Logic* 46:675–702. ISSN 0022-3611, 1573-0433. doi: 10.1007/s10992-016-9419-5.

—. 2017d. "Truthmaker Semantics." In *A Companion to the Philosophy of Language*, 556–577. John Wiley & Sons, Ltd. ISBN 978-1-118-97209-0. doi: 10.1002/9781118972090.ch22.

—. 2018a. "Compliance and Comand I—Categorical Imperatives." *The Review of Symbolic Logic* 11:609–633. ISSN 1755-0203, 1755-0211. doi: 10.1017/S175502031700020X.

—. 2018b. "Compliance and Command II, Imperatives and Deontics." *The Review of Symbolic Logic* 11:634–664. ISSN 1755-0203, 1755-0211. doi: 10.1017/S1755020318000059.

—. 2020. "Yablo on Subject-Matter." *Philosophical Studies* 177:129–171. ISSN 1573-0883. doi: 10.1007/s11098-018-1183-7.

—. Draft. "The World of Truthmaking." .

Fine, Kit and Jago, Mark. 2019. "Logic for Exact Entailment." *The Review of Symbolic Logic* 12:536–556. ISSN 1755-0203, 1755-0211. doi: 10.1017/S1755020318000151.

Fitting, Melvin. 1991. "Kleene's Logic, Generalized." *Journal of Logic and Computation* 1:797–810. ISSN 0955-792X, 1465-363X. doi: 10.1093/logcom/1.6.797.

—. 2002. "Bilattices Are Nice Things." *Self-reference* 53–77.

Ginsberg, Matthew L. 1988. "Multivalued Logics: A Uniform Approach to Inference in Artificial Intelligence." *Computational Intelegence* 4:265–316.

Mobasher, B., Pigozzi, D., Slutzki, G., and Voutsadakis, G. 2000. "A Duality Theory for Bilattices." *algebra universalis* 43:109–125. ISSN 0002-5240, 1420-8911. doi: 10.1007/s000120050149.

Rivieccio, Umberto. 2010. "An Algebraic Study of Bilattice-Based Logics." *arXiv:1010.2552 [math]* .

Van Fraassen, Bas C. 1969. "Facts and Tautological Entailments." *The Journal of Philosophy* 66:477–487. ISSN 0022-362X. doi: 10.2307/2024563.